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THE

CAMBRIDGE AND DUBLIN

MATHEMATICAL JOURNAL.

ON THE MATHEMATICAL THEORY OF ELECTRICITY IN EQUILIBRIUM.

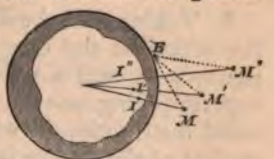
VI.—GEOMETRICAL INVESTIGATIONS REGARDING SPHERICAL CONDUCTORS.

By William Thomson.

Insulated Sphere subject to the Influence of a Body of any form Electrified in any given manner.

1. The problem of determining the distribution of electricity upon a sphere, or upon internal or plane spherical conducting surfaces, under the influence of an electrical point, was fully solved in Nos. IV. and V. of this series of papers. On the principle of the superposition of electrical forces (II. § 3) we may apply the same method to the solution of corresponding problems with reference to the influence of any number of given electrical points.

2. Thus let M, M', M'' be any number of electrical points possessing respectively m, m', m'' units of electricity, at distances f, f', f'' from C the centre of a sphere insulated and charged with a quantity Q of electricity. The actual distribution of electricity on the spherical surface must be such that the force due to it at any internal point shall be equal and opposite to the force due to the electricity at M, M', M''. Now if there were a distribution of electricity on the spherical surface such that the density at any point E would be $\frac{\lambda}{ME^3}$, the force due to this at any internal point would (IV. § 2) be the same as that due to a quantity $\frac{\lambda . 4\pi a}{f^2 - a^2}$



concentrated at the point M; and therefore if we take

$$\lambda = -\frac{(f^2 - a^2)m}{4\pi a},$$

the force at internal points due to this distribution would be equal and opposite to the force due to the actual electricity of M. We might similarly express distributions which would respectively balance the actions of M', M'', &c. upon points within the sphere; and thence, by supposing all those distributions to coexist on the surface, we infer that a single distribution such that the density at E is equal to

$$-\left\{\frac{(f^2 - a^2)m}{4\pi a}\frac{1}{ME^3} + \frac{(f'^2 - a^2)m'}{4\pi a}\frac{1}{M'E^3} + \frac{(f''^2 - a^2)m''}{4\pi a}\frac{1}{M''E^3}\right\}$$

would balance the joint action of all the electrical points M, M', M'', on points within the sphere. Again, from § 4 of the same article (No. IV.), we infer that the total quantity of electricity in such a distribution is

$$-\left(\frac{a}{f}m + \frac{a}{f'}m' + \frac{a}{f''}m''\right).$$

Hence, unless the data chance to be such that Q is equal to this quantity, a supplementary distribution will be necessary to constitute the actual distribution which it is required to find. The amount of this supplementary distribution will be

$$Q + \frac{a}{f}m + \frac{a}{f'}m' + \frac{a}{f''}m'';$$

which must be so distributed as to produce no force on internal points.

3. Taking then the distribution found above, which balances the action of the electricity at M, M', &c. on points within the sphere, and a uniform supplementary distribution; and superimposing one on the other, we obtain a resultant electrical distribution in which the density at any point E of the surface of the sphere is given by the equation

$$\rho = -\left\{\frac{(f^2 - a^2)m}{4\pi a}\frac{1}{ME^3} + \frac{(f'^2 - a^2)}{4\pi a}\frac{1}{M'E^3} + \&c.\right\}$$
$$+ \frac{Q + \frac{a}{f}m + \frac{a}{f'}m' + \&c.}{4\pi a^2} \quad \ldots\ldots\ldots(1);$$

and we draw the following conclusions:

to the other similar terms of (2). Again, the last term,

$$\frac{Q + \frac{a}{f} m + \frac{a}{f'} m' + \&c.}{4\pi a^2}$$

is the expression for the force at E, due to the imaginary electric point C, divided by 4π; and this force also is in the direction of the normal. Hence, with reference to the total resultant action at E, due to M, M', &c. and the spherical surface or the imaginary electrical points within it, we infer

(1) that this force is in the direction of the normal;

(2) that if R be its magnitude considered as positive or negative according as it is from or towards the centre of the sphere, and ρ the electrical density at E, we have

$$\rho = \frac{1}{4\pi} R \dots\dots\dots\dots\dots\dots (2).$$

These two propositions constitute the expression, for the case of a spherical conductor subject to any electric influence, of *Coulomb's Theorem.**

6. The total action exerted by the given electrical points, and by the sphere with its electricity disturbed by their influence, upon a given electrified body placed anywhere in their neighbourhood, might, as we have seen, be found by substituting in place of the sphere the group of electrical points which represents its external action, provided there were no disturbance produced by the influence of this electrified body. This hypothesis however cannot be true unless the sphere, after experiencing as a conductor the influence of M, M', &c., were to become a nonconductor so as to preserve with *rigidity* the distribution of its electricity when the new electrified body is brought into its neighbourhood: and consequently, when it is asserted that the resultant force *at* any external point P is due to the group of electrical points determined in the preceding paragraphs, we must remember that the disturbing influence that would be actually exerted upon the distribution on the spherical surface by a unit of electricity at the point P, is excluded in the definition (II. § 5) of the expression "*the resultant electrical force at a point.*"

* For a general demonstration of this theorem, virtually the same as th original demonstration given by Coulomb himself, see *Cambridge Matl Journal*, (1842) vol. III. p. 75.

7. The actual force exerted upon any one, M, of the influencing points may be determined by investigating the resultant force at M, due to all the others and to the conductor, and multiplying it by the quantity of electricity, m, situated at this point, since in this case the influence of the body on which the force is required has been actually taken into account.

8. It follows that the entire mutual action between all the given electrical points and the sphere under their influence is the same as the mutual action between the two systems of electrical points,

$$\left.\begin{array}{l} m \text{ at } M \\ m' \text{ at } M' \\ \cdots\cdots \\ \cdots\cdots \\ \\ \end{array}\right\} \text{ and } \left\{\begin{array}{ll} -\dfrac{a}{f}\,m & \text{at } I \\ -\dfrac{a}{f'}\,m' & \text{at } I' \\ \cdots\cdots\cdots\cdots & \cdots\cdots \\ \cdots\cdots\cdots\cdots & \cdots\cdots \\ Q + \dfrac{a}{f}\,m + \dfrac{a}{f'}\,m' + \&c. & \text{at } C. \end{array}\right.$$

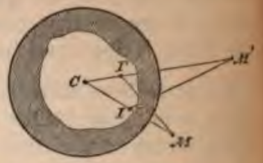
This action may be fully determined with any assigned *data*, by the elementary principles of statics.

9. There is a remarkable characteristic of this resultant action which ought not to be passed over, as it is related to a very important physical principle of symmetry, of which many other illustrations occur in the theories of electricity and magnetism. It is expressed in the following proposition:

The mutual action between a spherical conductor and any given electrified body consists of a single force in a line through the centre of the sphere.

Let us conceive the given electrified body either to consist of a group of electrical points, or to be divided into infinitely small parts, each of which may be regarded as an electrical point. The mutual action between the given body and the conducting sphere under its influence is therefore to be found by compounding all the forces between the points M, M', &c. of the given body, and the points I, I', &c.. . . . and C of the imaginary system within the sphere determined by the construction and formulæ of the preceding paragraphs. Of these, the forces between M and C, between M' and C', &c.; and again, between M and I, between M' and I', &c., are actually in lines passing through C, and therefore if there were no other forces to be taken into account the proposition would be proved. But we have also a set of forces between

M and I', between M and I'', &c., none of which, except in particular cases, are in lines through C, and therefore it remains for us to determine the nature of the resultant action of all these forces. For this purpose let us consider any two points M, M' of the given influencing body and the corresponding imaginary points I, I'; and let us take the force between M and I', and along with it the force between M' and I. These



two forces lie in the plane MCM', since, by the construction given above, I and I' are respectively in the lines CM and CM'; and hence they have a single resultant. Now the force in MI' is due to m units of electricity at M, and $-\frac{a}{f'}m'$ units at I'; and (II. § 4) it is therefore a force of repulsion equal to

$$\frac{-\frac{a}{f'}m'.m}{I'M^2}, \text{ or a force of attraction equal to } \frac{\frac{a}{f'}m'.m}{I'M^2}.$$

Similarly, we find $$\frac{\frac{a}{f}m.m'}{IM'^2}$$

for the attraction between M and I. Now since, by construction, $CM.CI=CM'.CI'$, the triangles $I'MC$, $IM'C$, which have a common angle at C, are similar. Hence

$$\frac{I'M^2}{IM'^2}=\frac{CI'.CM}{CI.CM'}=\frac{\frac{a^2}{f'}.f}{\frac{a^2}{f}.f'},$$

from which we deduce

$$\frac{\frac{a}{f'}m'.m}{I'M^2}f'=\frac{\frac{a}{f}m.m'}{IM'^2}f.$$

Now if we multiply the first member of this equation by $\sin CMI'$, we obtain the moment round C of the force between I' and M; and similarly, by multiplying the second member by $\sin CM'I$, we find the moment of the force between M' and I; and, since the angle at M is equal to the angle at M', we infer that the moments of the two forces

round C are equal. From this it follows that the resultant of the forces in MI' and $M'I$ is a force in a line passing through C. Now the entire group of forces between points of the given body and *non-correspondent* imaginary points, consists of pairs such as that which we have just been considering; and therefore the mutual action is the resultant of a number of forces in lines passing through C. This, compounded with the forces between M, M', &c. and the *corresponding* imaginary points, and the forces between M, M', &c. and the imaginary electrical point at C, gives for the total mutual action, a final resultant in a line passing through C.

10. It follows from this theorem that if a spherical conductor be supported in such a manner as to be able to turn freely round its centre, or round any axis passing through its centre, it will remain in equilibrium when subjected to the influence of any external electrified body or bodies. We may arrive at the same conclusion by merely considering the perfect symmetry of the sphere, round its centre or round any line through its centre, without assuming any specific results with reference to the distribution of electricity on spherical conductors. For if there were a tendency to turn round any diameter through the influence of external electrified bodies, the sphere would, on account of its symmetry, experience the same tendency when turned into any other position, its centre and the influencing bodies remaining fixed; and there would therefore result a continually accelerated motion of rotation. This being a physical impossibility, we conclude that the sphere can have no tendency to move when its centre is fixed, whatever be the electrical influence to which it is subjected.

11. It is very interesting to trace the different actions which, according to the synthetical solution of the problem of electrical influence investigated above, must balance to produce this equilibrium round the centre of a spherical conductor subjected to the influence of a group of electrical points. Let us for example consider the case of two influencing points.

For fixing the ideas, let us conceive the sphere to be capable of turning round a vertical axis, and let the influencing points be situated in the horizontal plane of its centre, C. If at first there be only one electrical point, M, which we may suppose to be positive, the sphere under its influence will be electrified with a distribution symmetrical round the line MC, but with more negative, or as the case

may be, less positive, electricity, on the hemisphere of the surface next M than on the remote hemisphere. If another positive electrical point, M', be brought into the neighbourhood of the sphere, on a level with its centre, and on one side or the other of MC, and if for a moment we conceive the sphere to be a perfect nonconductor of electricity; this second point, acting on the electricity as distributed under the influence of the first, will make the sphere tend to turn round its vertical axis. Thus if AA_1 be a diameter of the sphere in the line $MACA_1$, the sphere would tend to turn from its primitive position so as to bring the point A of its surface nearer M'. If now the sphere be supposed to become a perfect conductor, the distribution of its electricity will be altered so as to be no longer symmetrical round AA_1. This alteration we may conceive to consist of the superposition of a distribution of equal quantities of positive and negative electricity symmetrically distributed round the line $M'C$, with the negative electricity preponderating on the hemisphere nearest to M'. To obtain the total action of the two points on the electrified sphere, it will now be necessary to compound the action of M', and the action of M, on this superimposed distribution with the action previously considered. Of these the former consists of a simple force of attraction in the line $M'C$; but the latter, if referred to C the centre of the sphere, will give, besides a simple force, a couple round a vertical axis, tending to turn the sphere in such a direction as to bring the point A' of its surface nearer M. Now, as we know *a priori* that there can be no resultant tendency to turn arising from the entire action upon the sphere, it follows that the moment of this couple must be equal to the moment of the contrary couple, which, as we have seen previously, results from the action of M' on the sphere as primitively electrified under the influence of M. This is precisely the proposition of which a synthetical demonstration was given in § 9, and we accordingly see that that demonstration is merely the verification of a proposition of which the truth is rendered certain by *a priori* reasoning founded on general physical principles.

12. When the influencing body, instead of being as we have hitherto conceived it a finite group of isolated electrical points, is a continuous mass continuously electrified, we must imagine it to be divided into an infinite number of electrical points; and then, by means of the integral calculus,

the expressions investigated above may be modified so as to be applicable to any conceivable case.

13. It appears from the considerations adduced in No. v. that it is impossible to have an internal spherical conducting surface, or an infinite plane conducting surface, insulated and charged with a given amount of electricity; and that consequently, there being no "uniform supplementary distributions" to be taken into account, the solutions of ordinary problems with reference to such surfaces are somewhat simpler than those in which it may be proposed to consider an insulated conducting sphere possessing initially a given electrical charge. All the investigations of the present article, except those which have reference to the "supplementary distribution" and are not required, are at once applicable to cases of internal or of plane conducting surfaces.

14. The importance of considering the imaginary electrical points I, I', &c. (and C, in the case of an external spherical surface), whether for solving problems with reference to the mutual forces called into action by the electrical excitation, or for determining the distribution of electricity on the spherical surface, has been shewn in what precedes. Hence it will be useful, before going farther in the subject, to examine the nature of such groups of imaginary points, when the influencing bodies are either finite groups of electrical points, or continuously electrified bodies.

15. The term *Electrical Images*, which will be applied to the imaginary electrical points or groups of electrical points, is suggested by the received language of Optics; and the close analogy of optical images will, it is hoped, be considered as a sufficient justification for the introduction of a new and extremely convenient mode of expression into the Theory of Electricity.

Stockholm, September 20, 1849.

ON CERTAIN THEOREMS IN THE CALCULUS OF OPERATIONS.

By W. F. DONKIN, M.A., Savilian Professor of Astronomy in the University of Oxford.

[Some of the following theorems may be regarded as generalizations of some of those given by Mr. Bronwin in a recent number of the *Journal*. It is proper therefore to state that those marked (6), (7), (8), (9), occurred to me before I was acquainted with his paper.]

Let ϖ and ρ be symbols of operation, and let

$$\varpi\rho u - \rho\varpi u = \rho_1 u,$$

or dropping the subject u, let

$$\left.\begin{array}{l} \varpi\rho \ - \rho\varpi \ = \rho_1 \\ \varpi\rho_1 - \rho_1\varpi = \rho_2 \\ \cdots\cdots\cdots\cdots \\ \varpi\rho_n - \rho_n\varpi = \rho_{n+1} \end{array}\right\} \qquad (1).$$

Then if $f(x)$ be a function capable of development in positive or negative powers of x, the following equations are true, viz.

$$f(\varpi)\rho = \rho f(\varpi) + \frac{\rho_1}{1} f'(\varpi) + \frac{\rho_2}{1.2} f''(\varpi) + \ldots\ldots (2),$$

$$\rho f(\varpi) = f(\varpi)\rho - f'(\varpi)\frac{\rho_1}{1} + f''(\varpi)\frac{\rho_2}{1.2} - \ldots\ldots (3).$$

Also

$$f\left(\varpi + \frac{1}{\rho}\,\rho_1\right) = f(\varpi) + \frac{1}{\rho}\,\rho_1 f'(\varpi) + \frac{1}{\rho}\rho_2\frac{f''(\varpi)}{1.2} + \frac{1}{\rho}\rho_3\frac{f'''(\varpi)}{1.2.3} + \ldots\ldots (4),$$

$$f\left(\varpi - \rho_1\frac{1}{\rho}\right) = f(\varpi) - f'(\varpi)\,\rho_1\frac{1}{\rho} + \frac{f''(\varpi)}{1.2}\,\rho_2\frac{1}{\rho} - \frac{f'''(\varpi)}{1.2.3}\rho_3\frac{1}{\rho} + \ldots\ldots (5).$$

The demonstrations of these theorems are very simple; but before exhibiting them it will be as well to give some illustrative examples of their use.

Let D denote $\dfrac{d}{dx}$, and let $X, X', X'', \ldots$ represent a function of x and its successive derived functions. Put $\varpi = D$ and $\rho = X$; then $\rho_1 u = DXu - XDu = X'u$ or $\rho_1 = X'$, and similarly $\rho_2 = X''$, $\rho_3 = X'''$, ... and the series above

written (in which it must be remembered that the subject is omitted) become

$$f(D)\,X = Xf(D) + Xf'(D) + \frac{X''}{1.2}f''(D) + \ldots\ldots\ldots (6),$$

$$Xf(D) = f(D)\,X - f'(D)\,X' + f''(D)\frac{X''}{1.2} - \ldots\ldots\ldots (7),$$

$$f\left(D + \frac{X'}{X}\right) = f(D) + \frac{X'}{X}f'(D) + \frac{X''}{X}\frac{f''(D)}{1.2} + \ldots\ldots\ldots (8),$$

$$f\left(D - \frac{X'}{X}\right) = f(D) - f'(D)\frac{X'}{X} + \frac{f''(D)}{1.2}\frac{X''}{X} - \ldots\ldots\ldots (9).$$

Again, let u_x, v_x be functions of x. Put $\varpi = e^D$, $\rho = v_x$, and let u_x be the subject. We have then

$$(\varpi\rho - \rho\varpi)\,u_x = v_{x+1}\,u_{x+1} - v_x\,u_{x+1} = \Delta v_x \,.\, e^D u_x,$$

hence $\rho_1 = \Delta v_x \,.\, e^D$, and similarly $\rho_2 = \Delta^2 v_x \,.\, e^{2D}, \ldots$

Hence, from (2) and (3),

$$f(e^D).v_x u_x = v_x f(e^D)\,u_x + \frac{\Delta v_x}{1}.f'(e^D)\,u_{x+1} + \frac{\Delta^2 v_x}{1.2}f''(e^D).u_{x+2} + \ldots (10),$$

$$v_x f(e^D)\,u_x = f(e^D).v_x u_x - f'(e^D).\frac{\Delta v_x.u_{x+1}}{1} + f''(e^D).\frac{\Delta^2 v_x.u_{x+2}}{1.2} - \ldots (11).$$

From which, by putting $f(e^D) = (e^D - 1)^n = \Delta^n$, or again $f(e^D) = e^{nD}$, several well-known developments may be derived.

Again, from (6), putting $X = x$, we obtain

$$f(D)\,x - xf(D) = f'(D),$$

and consequently $f'(D)\,x - xf'(D) = f''(D)$,

. . . .

In the general formulæ let $\varpi = x$, $\rho = \phi(D)$, and the equations just written give $\rho_1 = -\phi'(D)$, $\rho_2 = \phi''(D), \ldots\ldots$ and consequently, from (4) and (5),

$$f\left(x - \frac{\phi'(D)}{\phi(D)}\right) = f(x) - \frac{\phi'(D)}{\phi(D)}f'(x) + \frac{\phi''(D)}{\phi(D)}\frac{f''(x)}{1.2} - \ldots (12),$$

$$f\left(x + \frac{\phi'(D)}{\phi(D)}\right) = f(x) + f'(x)\frac{\phi'(D)}{\phi(D)} + \frac{f''(x)}{1.2}\frac{\phi''(D)}{\phi(D)} + \ldots (13).$$

The equations $f'(D) = f(D)\,x - xf(D)$,

$$f''(D) = f'(D)\,x - xf'(D),$$

. . . .

give, by successive substitution,

$$f^{(n)}(D) = f(D)x^n - nxf(D)x^{n-1} + \frac{n(n-1)}{1.2}x^2f(D)x^{n-2} - \ldots \pm x^n f(D);$$

and consequently, if δ denote differentiation *with respect to* D, and $\phi(\delta)$ be a function interpreted by development in positive powers of δ, it is easy to derive the following:

$$\phi(\delta)f(D) = f(D)\,\phi(x) - xf(D)\,\phi'(x) + \frac{x^2}{1.2}f(D)\,\phi''(x) - \ldots (14),$$

and, by writing the series for $f^{(n)}(D)$ in the inverse order,

$$\phi(-\delta)f(D) = \phi(x)f(D) - \phi'(x)f(D)\,x + \phi''(x)f(D)\frac{x^2}{1.2} - \ldots (15).$$

As an instance of verification, let $\phi(x) = \sin x$, and $f(D) = \frac{1}{D}$, then

$$\phi(-\delta)f(D) = -\sin\delta\,\frac{1}{D} = \frac{1}{2\sqrt{(-1)}}\left\{\frac{1}{D-\sqrt{(-1)}} - \frac{1}{D+\sqrt{-1}}\right\} = \frac{1}{D^2+1}.$$

Hence, from (15), we get

$$\frac{1}{D^2+1} = \sin x\,\frac{1}{D}\left(1 - \frac{x^2}{1.2} + \ldots\right) - \cos x\,\frac{1}{D}\left(x - \frac{x^3}{1.2.3} + \ldots\right);$$

or, restoring the subject,

$$\frac{1}{D^2+1}\,u = \sin x \int \cos x \,.\, u dx - \cos x \int \sin x \,.\, u dx,$$

which is the well-known interpretation of $\frac{1}{D^2+1}\,u$.

I now return to the proof of the fundamental equations (2), (3), (4), (5).

In the first place, we have

$$\left(\frac{1}{\rho}\,\varpi\rho\right)^2 = \frac{1}{\rho}\,\varpi\rho\,\frac{1}{\rho}\,\varpi\rho = \frac{1}{\rho}\,\varpi^2\rho\,;$$

and similarly, for all whole positive values of n,

$$\left(\frac{1}{\rho}\,\varpi\rho\right)^n = \frac{1}{\rho}\,\varpi^n\rho.$$

The same equation is true for negative values of n. For since

$$\frac{1}{\rho}\,\varpi\rho \,.\, \frac{1}{\rho}\,\frac{1}{\varpi}\,\rho = 1, \text{ we have } \frac{1}{\rho}\,\frac{1}{\varpi}\,\rho = \left(\frac{1}{\rho}\,\varpi\rho\right)^{-1}:$$

and since, by the preceding proposition,

$$\left(\frac{1}{\rho}\frac{1}{\varpi}\rho\right)^n = \frac{1}{\rho}\left(\frac{1}{\varpi}\right)^n\rho,$$

(n being positive), it follows that

$$\frac{1}{\rho}\varpi^{-n}\rho = \left(\frac{1}{\rho}\varpi\rho\right)^{-n}.$$

If, then, $f(x)$ be any function capable of development in positive or negative powers of x, the following equation is true:

$$f\left(\frac{1}{\rho}\varpi\rho\right) = \frac{1}{\rho}f(\varpi)\rho \qquad (16).$$

Also, from the first of equations (1), we have

$$\frac{1}{\rho}\varpi\rho = \varpi + \frac{1}{\rho}\rho_1,$$

$$\rho\varpi\frac{1}{\rho} = \varpi - \rho_1\frac{1}{\rho},$$

and consequently

$$\left.\begin{aligned}\frac{1}{\rho}f(\varpi)\,\rho &= f\left(\varpi + \frac{1}{\rho}\rho_1\right)\\ \rho\, f(\varpi)\,\frac{1}{\rho} &= f\left(\varpi - \rho_1\frac{1}{\rho}\right)\end{aligned}\right\} \qquad (17).$$

Particular cases of these last equations are the known theorems,

$$\frac{1}{X}f(D)\,X = f\left(D + \frac{X'}{X}\right), \quad Xf(D)\,\frac{1}{X} = f\left(D - \frac{X'}{X}\right).$$

Again, equations (1) give

$$\varpi\rho = \rho\varpi + \rho_1,$$

$$\varpi^2\rho = \varpi\rho\varpi + \varpi\rho_1 = (\rho\varpi + \rho_1)\varpi + \rho_1\varpi + \rho_2,$$

$$= \rho\varpi^2 + 2\rho_1\varpi + \rho_2:$$

and in like manner, for positive values of n,

$$\varpi^n\rho = \rho\varpi^n + n\rho_1\varpi^{n-1} + \frac{n(n-1)}{1.2}\rho_2\varpi^{n-2} + \ldots + n\rho_{n-1}\varpi + \rho_n \quad (18).$$

Similarly, beginning with $\rho\varpi = \varpi\rho - \rho_1$, we obtain

$$\rho\varpi^n = \varpi^n\rho - n\varpi^{n-1}\rho_1 + \frac{n(n-1)}{1.2}\varpi^{n-2}\rho_2 - \ldots \pm n\varpi\rho_{n-1} \mp \rho_n \quad (19).$$

The same formulæ are true for negative values of n. For first we have

$$\rho = \frac{1}{\varpi}\rho\varpi + \frac{1}{\varpi}\rho_1,$$

and therefore $$\frac{1}{\varpi}\rho = \rho\frac{1}{\varpi} - \frac{1}{\varpi}\rho_1\frac{1}{\varpi},$$

and similarly, $$\frac{1}{\varpi}\rho_1 = \rho_1\frac{1}{\varpi} - \frac{1}{\varpi}\rho_2\frac{1}{\varpi}:$$

$$\dots\dots\dots\dots\dots$$

hence by successive substitution we easily find

$$\frac{1}{\varpi}\rho = \rho\frac{1}{\varpi} - \rho_1\frac{1}{\varpi^2} + \rho_2\frac{1}{\varpi^3} - \rho_3\frac{1}{\varpi^4} + \dots\dots\dots (20),$$

which proves (18) for $n = -1$. Suppose then (m being positive)

$$\frac{1}{\varpi^m}\rho = \rho\frac{1}{\varpi^m} - m\rho_1\frac{1}{\varpi^{m+1}} + \frac{m(m+1)}{1.2}\rho_2\frac{1}{\varpi^{m+2}} - \dots,$$

it follows that

$$\frac{1}{\varpi^{m+1}}\rho = \frac{1}{\varpi}\rho\frac{1}{\varpi^m} - m\frac{1}{\varpi}\rho_1\frac{1}{\varpi^{m+1}} + \frac{m(m+1)}{1.2}\frac{1}{\varpi}\rho_2\frac{1}{\varpi^{m+2}} - \dots$$

Substitute for $$\frac{1}{\varpi}\rho,\quad \frac{1}{\varpi}\rho_1,\quad \frac{1}{\varpi}\rho_2 \dots,$$

their values given by (20), and by the similar series which might obviously be obtained for

$$\frac{1}{\varpi}\rho_1,\quad \frac{1}{\varpi}\rho_2,\dots,$$

and it will be found that the general term in the development of

$$\frac{1}{\varpi^{m+1}}\rho \text{ is } (-1)^r.\frac{(m+1)(m+2)\dots(m+r)}{1.2\dots r}.\rho_r\frac{1}{\varpi^{m+1+r}}.$$

Hence if the theorem be true for any one value of m, it is true for all greater values: but it has been proved for $m = 1$, and therefore it is true for all values.

The process for equation (19) would of course be exactly similar. Now (18) and (19) being established, the theorems expressed by (2) and (3) follow obviously, and (4) and (5) are immediately derived from these by the help of the formulæ (17).

The developments of $\frac{1}{\rho}\rho_n$ and $\rho_n\frac{1}{\rho}$ are remarkable. We have in the first place,

$$\rho_1 = \varpi\rho - \rho\varpi, \quad \rho_2 = \varpi\rho_1 - \rho_1\varpi = \varpi^2\rho - 2\varpi\rho\varpi + \rho\varpi^2,$$

and in general

$$\rho_n = {}^n\rho\varpi - n\varpi^{n-1}\rho\varpi + \frac{n(n-1)}{1.2}\varpi^{n-2}\rho\varpi^2 - \ldots \pm n\varpi\rho\varpi^{n-1} \mp \rho\varpi^n;$$

and therefore, putting

$$\frac{1}{\rho}\varpi\rho = \sigma, \quad \rho\varpi\frac{1}{\rho} = \tau,$$

and taking account of (16), we get

$$\frac{1}{\rho}\rho_n = \sigma^n - n\sigma^{n-1}\varpi + \frac{n(n-1)}{1.2}\sigma^{n-2}\varpi^2 - \ldots \pm n\sigma\varpi^{n-1} \mp \varpi^n,$$

$$\rho_n\frac{1}{\rho} = \varpi^n - n\varpi^{n-1}\tau + \frac{n(n-1)}{1.2}\varpi^{n-2}\tau^2 - \ldots \pm n\varpi\tau^{n-1} \mp \tau^n,$$

so that $\frac{1}{\rho}\rho_n$, $\rho_n\frac{1}{\rho}$ are found by developing $(\sigma - \varpi)^n$, $(\varpi - \tau)^n$ as if the symbols were commutative, with the condition that in each term of the first, σ must precede ϖ, and in each term of the last ϖ must precede τ. It might perhaps be convenient to adopt the notation $(\sigma - \varpi)_n$, $(\varpi - \tau)_n$ for such developments. The theorems (4) and (5) might thus be written

$$f(\sigma) = f(\varpi) + (\sigma - \varpi)f'(\varpi) + \frac{(\sigma - \varpi)_2}{1.2}f''(\varpi) + \ldots\ldots \text{(21)},$$

$$f(\tau) = f(\varpi) - f'(\varpi).(\varpi - \tau) + f''(\varpi).\frac{(\varpi - \tau)_2}{1.2} - \ldots\ldots \text{(22)}.$$

And these formulæ are true, whatever be the operations denoted by ϖ, σ, τ; for, the two operations σ and ϖ being given, there must exist some operation ρ such as to verify the relation $\sigma = \frac{1}{\rho}\varpi\rho$. And a similar remark will apply to ϖ and τ. We may therefore write σ instead of τ in (22), and then the two series correspond exactly with the two following ways of writing Taylor's theorem:

$$f(a) = f(b) + (a - b)f'(b) + \frac{(a - b)^2}{1.2}f''(b) + \ldots,$$

$$f(a) = f(b) - f'(b).(b - a) + f''(b).\frac{(b - a)^2}{1.2} - \ldots,$$

with which (21) and (22) become identical whenever σ and ϖ represent commutative operations.

These two theorems (21) and (22), being in themselves more simple and general than (4) and (5), might naturally have been placed first; but their form is less convenient for illustration by examples, and therefore I did not wish to exhibit them apart from their demonstrations.

It seems to result from these investigations, and from the researches of Mr. Boole, Mr. Bronwin, and others, in the same department, that there is much more analogy than might have been expected *à priori* between the laws of commutative and non-commutative symbols. It appears probable also that in the further progress of inquiry some extension of notation may become necessary, or at least convenient. For example, it may become desirable to have some symbol analogous to the functional symbol f, to express the series

$$a_0 + a_1(\sigma - \varpi) + a_2(\sigma - \varpi)_2 + \ldots + a_n(\sigma - \varpi)_n + \ldots,$$

where $a_0, a_1, \ldots$ are the coefficients of the powers of x in the development of $f(x)$. As to the notation $(\sigma - \varpi)_n$ itself, I do not propose it as free from objection: but it is the best that occurs to me.

POSTSCRIPT.

Since writing the above, I have found that the principal theorems may be made rather more general, as follows:—

Let ϖ, ρ, σ be any three operations; then putting

$$\begin{aligned} \varpi\rho - \rho\sigma &= \rho_1, \\ \varpi\rho_1 - \rho_1\sigma &= \rho_2, \\ &\ldots\ldots\ldots\ldots \\ \varpi\rho_n - \rho_n\sigma &= \rho_{n+1}; \end{aligned}$$

and proceeding exactly as before, we find

$$\frac{1}{\rho} f(\varpi)\rho = f\left(\sigma + \frac{1}{\rho}\rho_1\right), \quad \rho f(\sigma)\frac{1}{\rho} = f\left(\varpi - \rho_1\frac{1}{\rho}\right),$$

also $$\frac{1}{\rho}\rho_n = \frac{1}{\rho}\varpi^n\rho - n\frac{1}{\rho}\varpi^{n-1}\rho.\sigma + \frac{n(n-1)}{1.2}\frac{1}{\rho}\varpi^{n-2}\rho.\sigma^2 - \ldots,$$

$$\rho_n\frac{1}{\rho} = \varpi^n - n\varpi.\rho\sigma\frac{1}{\rho} + \ldots;$$

or, $\left(\text{putting } \frac{1}{\rho}\varpi\rho = \varpi', \quad \rho\sigma\frac{1}{\rho} = \sigma'\right)$,

$$\frac{1}{\rho}\rho_n = (\varpi' - \sigma)_n, \quad \rho_n.\frac{1}{\rho} = (\varpi - \sigma')_n.$$

We have also, by successive substitution, as before,

$$\varpi^n \rho = \rho\sigma^n + n\rho_1\sigma^{n-1} + \frac{n(n-1)}{1.2}\rho_2\sigma^{n-2} + \ldots,$$

$$\rho\sigma^n = \varpi^n\rho - n\varpi^{n-1}\rho_1 + \frac{n(n-1)}{1.2}\varpi^{n-2}\rho_2 - \ldots,$$

and consequently

$$f(\varpi)\rho = \rho f(\sigma) + \rho_1 f'(\sigma) + \frac{\rho_2}{1.2}f''(\sigma) + \ldots,$$

$$\rho f(\sigma) = f(\varpi)\rho - f'(\varpi)\rho_1 + \frac{1}{1.2}f''(\varpi)\rho_2 - \ldots.$$

Whether any more interesting results can be obtained from these expressions than from the less general ones given above, I am not at present prepared to say. If however we put $\rho = 1$, they afford at once a better demonstration of the series analogous to Taylor's theorem. For we have then

$$\varpi' = \varpi, \quad \sigma' = \sigma, \quad \rho_n = (\varpi - \sigma)_n,$$

and the two series just written become

$$f(\varpi) = f(\sigma) + (\varpi - \sigma)f'(\sigma) + \frac{(\varpi-\sigma)_2}{1.2}f''(\sigma) + \ldots,$$

$$f(\sigma) = f(\varpi) - f'(\varpi)(\varpi - \sigma) + f''(\varpi)\frac{(\varpi-\sigma)_2}{1.2} - \ldots.^*$$

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* These two series may also be deduced independently thus:

$$f(\varpi) = f(0) + \frac{\varpi}{1}f'(0) + \frac{\varpi^2}{1.2}f''(0) + \ldots,$$
$$= f(\sigma - \sigma) + \frac{\varpi}{1}f'(\sigma - \sigma) + \frac{\varpi^2}{1.2}f''(\sigma - \sigma) + \ldots,$$
$$= f(\sigma) - \sigma f'(\sigma) + \frac{\sigma^2}{1.2}f''(\sigma) - \ldots,$$
$$+ \frac{\varpi}{1}\{f'(\sigma) - \sigma f''(\sigma) + \ldots\},$$
$$+ \frac{\varpi^2}{1.2}f''(\sigma) + \ldots,$$
$$= f(\sigma) + (\varpi - \sigma)f'(\sigma) + \frac{\varpi^2 - 2\varpi\sigma + \sigma^2}{1.2}f''(\sigma) - \ldots,$$

which is the first series. And the other may be obtained in the same way from

$$f(\varpi) = f(0) + f'(0)\varpi + \frac{1}{1.2}f''(0)\varpi^2 + \ldots,$$

and

$$f(0) = f(\sigma) - f'(\sigma)\sigma + \frac{1}{1.2}f''(\sigma)\sigma^2 - \ldots,$$
$$f'(0) = f'(\sigma) - f''(\sigma)\sigma + \ldots.$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

It is to be remarked that the law by which $(\varpi - \sigma$ deduced from $(\varpi - \sigma)_n$, namely

$$(\varpi - \sigma)_{n+1} = \varpi(\varpi - \sigma)_n - (\varpi - \sigma)_n\,\sigma,$$

is identical with the index-law when ϖ and σ are co tative; as of course it ought to be, since in that case

$$(\varpi - \sigma)_n = (\varpi - \sigma)^n.$$

April, 1849.

ON CURVES OF DOUBLE CURVATURE AND DEVELOP SURFACES.*

By Arthur Cayley.

Some very elegant researches are to be found in] "Theorie der algebraischen Curven," with referen number of singularities (points of inflexion, double &c.) of plane curves. The same principles are appl the case of curves of double curvature. Consider purpose a series of points in space, the lines passing two consecutive points, and the planes passing throu consecutive points; or what is the same thing, a lines, each of which intersects the consecutive points of intersection of two consecutive lines, and t containing these pairs of lines; or again, a series c the lines of intersection of two consecutive planes, points of intersection of three consecutive plane a system may be termed a "simple system." Th is evidently formed of a curve of double curvatu developable surface, the curve is the edge of reg the surface, the surface is the developable osculat curve. The points of the system are points upon t the lines of the system are tangents of the curve, t of the system are osculating planes of the curve. planes of the system are tangent planes of the su lines of the system are generating lines of the su points of the system are what may be termed poi gression of the surface. The terms '*line through t*

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* Translated with some slight alterations from a Memoir ir Journal, tom. x. p. 245-250.

'*line in two planes*,' will be employed in the sequel to denote a line drawn through any two points (not in general consecutive) of the system, and a line of intersection of any two planes (not in general consecutive) of the system. Again, each line of the system is in general intersected by a certain [illegible] lines (not consecutive) of the system. The point [illegible]ion of such a pair of lines will be termed *a point* [illegible], and the plane which contains such a pair of lines [illegible]*ough two lines*.

[illegible] that a given plane contains in general m points of [illegible], that a given line meets in general r lines of the [illegible]d that a given point is situated in general in n [illegible]he system. The system is said to be of the order [illegible]k r, and the class n. The order of the curve is [illegible]he order of the system or to m; the class of the [illegible]ual to the rank of the system or to r. And in like [illegible]e order of the surface is equal to the rank of the [illegible] to r, the class of the surface is equal to the class of [illegible] or to n.

[illegible]ing premised, the proper singularities (see the work [illegible]erred to) are the two following, analogous to the [illegible] of inflexion and cusps of a plane curve.

[illegible]en four consecutive points lie in the same plane, [illegible]s the same thing, when three consecutive lines lie [illegible]ne plane, or again, when two consecutive planes [illegible]—there is what I term *a stationary plane*, and the [illegible]f these planes will be represented by the letter α.

[illegible]en four consecutive planes meet in the same point, [illegible] the same thing, when three consecutive lines meet [illegible]ne point, or again, when two consecutive points [illegible]—there is what I term *a stationary point*, and the [illegible]f these points will be represented by the letter β.

[illegible] former case there is a point of spherical inflexion [illegible] curve, and a line of inflexion upon the surface; but [illegible]ter case the singularity has not had any name given [illegible] there is no need to supply this defect, as the nature [illegible]ngularity in question is sufficiently defined by the [illegible]*ionary point*. The two cases mentioned above may [illegible]l the simple singularities of a system. There are of [illegible]ngularities of a higher order, but these do not require [illegible]nded to here.

[illegible]re moreover singularities of a different species [illegible] in some measure to the double points and double

tangents of a plane curve, but having relation to an indeterminate point or plane out of the system: in fact,

3. A given plane may contain *in general* a number g of *lines in two planes.*

4. A given point may lie *in general* upon a number h of *lines through two points.*

5. A given plane may contain *in general* a number x of *points in two lines.*

6. A given point may lie *in general* upon a number y of *planes through two lines.*

These four cases are the simple improper singularities of the system.

It is required to determine the relations which exist between the quantities $m, r, n, a, \beta, g, h, x, y$.

In order to effect this it will be convenient to quote Plücker's formulæ for plane curves (p. 211): but in order to avoid confusion the letters have been changed.

If μ denote the order of a curve, ν its class, δ the number of double points, κ that of the cusps, τ the number of double tangents, ι that of the points of inflexion, then these quantities are connected by the equations,

$$\nu = \mu(\mu - 1) - (2\delta + 3\kappa),$$

$$\iota = 3\mu(\mu - 2) - (6\delta + 8\kappa),$$

$$\tau = \tfrac{1}{2}\mu(\mu - 2)(\mu^2 - 9) - (2\delta + 3\kappa)\{\mu(\mu - 1) - 6\} + 2\delta(\delta - 1) + \tfrac{9}{2}\kappa(\kappa - 1) + 6\delta\kappa,$$

$$\mu = \nu(\nu - 1) - (2\tau + 3\iota),$$

$$\kappa = 3\nu(\nu - 2) - (6\tau + 8\iota),$$

$$\delta = \tfrac{1}{2}\nu(\nu - 2)(\nu^2 - 9) - (2\tau + 3\iota)\{\nu(\nu - 1) - 6\} + 2\tau(\tau - 1) + \tfrac{9}{2}\iota(\iota - 1) + 6\tau\iota;$$

the last three of which are derivable from the first three, and *vice versâ.**

Consider now a given plane in conjunction with the system. This plane cuts the surface in a plane curve. The points of this curve are the points of intersection of the plane with the

* [These formulæ may also be conveniently used in the following shape:

$$\iota - \kappa = 3(\nu - \mu),$$

$$2(\tau - \delta) = (\nu - \mu)(\nu + \mu - 9).\text{—G. S.]}$$

lines of the system, the tangents of the curve are the lines of intersection of the plane with the planes of the system. It is clear that the curve is of the order r and of the class n. Every time that the plane contains a point in two lines there is a double point upon the curve; at the points where the plane meets the curve of double curvature there is a cusp upon the curve, (for in this case there are two lines of the system which cut the plane in the same point, that is to say, there is upon the curve of intersection a stationary point or cusp). When the plane contains a line in two planes there is a double tangent to the curve; and finally, for every stationary plane of the system there is a stationary tangent or inflexion in the curve. We have therefore a curve of the order r and the class n with x double points, m cusps, g double tangents, and a points of inflection; and we thus obtain the six equations (equivalent to three independent equations),

$$n = r(r-1) - (2x + 3m),$$

$$a = 3r(r-2) - (6x + 8m),$$

$$g = \tfrac{1}{2}r(r-2)(r^2-9) - (2x+3m)\{r(r-1)-6\} + 2x(x-1)$$
$$+ \tfrac{9}{2}m(m-1) + 6xm.$$

$$r = n(n-1) - (2g + 3a),$$

$$m = 3n(n-2) - (6g + 8a),$$

$$x = \tfrac{1}{2}n(n-2)(n^2-9) - (2g+3a)\{n(n-1)-6\} + 2g(g-1)$$
$$+ \tfrac{9}{2}a(a-1) + 6ga.$$

Similarly, in considering a given point in conjunction with the system, this point and the curve of double curvature determine a conical surface; and by reasoning similar to what has preceded it may be shewn that this conical surface is of the order m and the class r, and has h double lines, β cusped lines, y double tangent planes, and n lines of inflexion: we thus obtain the following six equations (equivalent to three independent equations),

$$r = m(m-1) - (2h + 3\beta),$$

$$n = 3m(m-2) - (6h + 8\beta),$$

$$y = \tfrac{1}{2}m(m-2)(m^2-9) - (2h+3\beta)\{m(m-1)-6\} + 2h(h-1)$$
$$+ \tfrac{9}{2}\beta(\beta-1) + 6h\beta;$$

$$m = r(r-1) - (2y + 3n),$$

$$\beta = 3r(r-2) - (6y + 8n),$$

$$h = \tfrac{1}{2}r(r-2)(r^2-9) - (2y+3n)\{r(r-1)-6\} + 2y(y-1) + \tfrac{9}{2}n(n-1) + 6yn;$$

where, comparing these with the preceding six equations, we may remark the correspondence between the symbols $m, r, n, \alpha, \beta, g, h, x, y$, and $n, r, m, \beta, \alpha, h, g, y, x$.

Considering a curve of double curvature of a given order m, the values of h and β are (within certain limits) arbitrary, and we then have to determine the other quantities, the following equations,*

$$r = m(m-1) - (2h + 3\beta),$$

$$n = 3m(m-2) - (6h + 8\beta),$$

$$y = \tfrac{1}{2}m(m-2)(m^2-9) - (2h+3\beta)\{m(m-1)-6\} + 2h(h-1) + \tfrac{9}{2}\beta(\beta-1) + 6h\beta,$$

$$x = \tfrac{1}{2}m(m-1)(m^2-m-4) + \tfrac{1}{2}(2h+3\beta)(2h+3\beta+1) - m(m-1)(2h+3\beta) + 3h + 4\beta,$$

$$\alpha = 2m(3m-7) - 3(4h + 5\beta),$$

$$g = \tfrac{1}{2}m(3m-7)(3m^2-5m-7) + \tfrac{1}{2}(6h+8\beta)(6h+8\beta+1) - 3m(m-2)(6h+8\beta) + 19h + 24\beta.\dagger$$

In the particular case of a plane system the last three equations cease to have any meaning, but the first three continue true. It is hardly necessary to remark, that for the system which is the reciprocal of the given system it is only necessary to change $m, r, n, \alpha, \beta, g, h, x, y$ into $n, r, m, \beta, \alpha, h, g, y, x$.

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* The last three of these equations differ in form from the corresponding equations in Liouville, which do not give explicitly the values of x, α, g in terms of m, h, β. There is an obvious misprint in the fourth equation in Liouville—the letter h should be replaced by n.

† [To these formulæ we may add the following analogous to these given in the note on last page,

$$\alpha - \beta = 2(n-m),$$

$$x - y = (n-m),$$

$$2(g-h) = (n-m)(n+m-7).\text{—G. S.]}$$

ON THE CLASSIFICATION OF CURVES OF DOUBLE CURVATURE.

By the Rev. George Salmon.

1. The theory of curves of double curvature is attended with some difficulty, arising from the fact that our analytical methods are not applicable to them directly, but only through the intervention of surfaces. Thus a curve of double curvature is in general expressed by the equations of two surfaces which pass through it,

$$\phi(x, y, z) = 0, \quad \psi(x, y, z) = 0:$$

but it will in general happen that these equations will express not only the curve we wish to investigate, but also an extraneous curve with which we are not concerned. This inconvenience may be illustrated by the example of a conic section in space expressed by its two projections,

$$\phi(x, z) = 0, \quad \psi(y, z) = 0.$$

These are the equations of two cylinders which intersect not only in the given conic but also in a second plane curve, with which we are not supposed to be concerned. In this example it is true, the equations admit of being transformed into an unambiguous form, and the conic may be expressed by means of the equation of its plane, and that of some surface of the second degree passing through it. In the general case, however, such a transformation will not be possible. In other words, it is not generally true that a curve of double curvature may be considered as the complete intersection of two surfaces, and it is therefore impossible (except in particular cases) to find two equations which shall represent a given curve, without at the same time denoting some extraneous curve.

2. No ambiguity however occurs when we express analytically, not the curve itself, but the developable surface formed by its tangents. The curve, which is the edge of regression of this surface, is then completely determined. When we proceed however to the analytical solution of the problem "to find the edge of regression of a developable whose equation is given," we are met again by the difficulty noticed in the last paragraph. These difficulties are considerably mitigated by a memoir of Mr. Cayley's, published in Liouville's Journal, in which, without any consideration of surfaces, he obtains several important relations connecting the curve and its developable; so that being given a section of a developable, we are able to tell the properties of the curve

which is its edge of regression. As I shall have constant occasion in what follows to make use of Mr. Cayley's results, I have requested him to furnish me with a translation of his memoir to be prefixed to the present paper. The immediate object of this article is to take the first steps toward a classification of curves of double curvature considered as the intersection of surfaces; commencing with the simplest curves to examine of each in succession, whether it be the complete intersection of two surfaces, and what are the two simplest species of surfaces which can be drawn through it; and reciprocally commencing with the smallest values of m and n, to inquire what are the different curves which can form part of the intersection of two surfaces of the m^{th} and n^{th} degrees. Before proceeding to this examination I shall give some results obtained without any consideration of surfaces, and which may be considered as supplemental to the preceding translation from Liouville. It is right to mention that the present investigation having been pursued in conjunction with Mr. Cayley, our results are for the most part common property, and are now published with his consent.

3. Mr. Cayley has given (p. 21) the number of singular points and lines in a section of the developable surface, and in a cone standing on the given curve. We proceed now to examine how these numbers are modified for the particular cases where the section is made by a plane of the system, and where the vertex of the cone is a point of the system. In this case we shall find

$$\mu = r - 2, \quad \iota = a, \quad \tau = g - n + 2,$$
$$\nu = n - 1, \quad \kappa = m - 3, \quad \delta = x - 2r + 8.$$

These values being connected with each other by the relations

$$\iota - \kappa = 3(\nu - \mu),$$
$$2(\delta - \tau) = (\mu - \nu)(\mu + \nu - 9),$$

it would be sufficient to establish any three of them.

Now since every plane of the system touches the developable along the whole length of the corresponding line of the system, the section by such a plane will be two coincident right lines and a curve for which $\mu = r - 2$.

The class of this curve is determined (p. 21) by the number of planes of the system which can be drawn through any point of the section: and since in this case one of the planes coincides with the plane of the section itself, the number of remaining planes gives $\nu = n - 1$.

The investigation of the points of inflection is unaltered: we have therefore, as at p. 21, $\iota = \alpha$.

Since every plane of the system passes through 3 consecutive points of the system, such a plane can only contain $m - 3$ other points of the system: for the curve therefore which we are considering $\kappa = m - 3$; the point where that curve is touched by the corresponding right line of the system counting for three cusps in the complex section made up of the curve and that right line.

Each point where the curve part of the section is met by the line of the system counts for two double points on the total complex section; and since that right line touches the curve, it only intersects the curve in $r - 4$ other points. The number of double points is less therefore than in the general case by $2r - 8$.

4. The corresponding investigation for the cone whose vertex is a point of the system, gives

$$\mu = m - 1, \qquad \nu = r - 2,$$
$$\iota = n - 3, \qquad \kappa = \beta,$$
$$\tau = y - 2r + 8, \quad \delta = h - m + 2.$$

From these and the preceding values we learn that as an arbitrary plane contains g "lines in *two* planes," so a plane of the system contains $g - n + 2$ "lines in *three* planes." In like manner, through an arbitrary point can be drawn h "lines through *two* points," but through a point of the system $h - m + 2$ "lines through *three* points." There will in general be a finite number of "lines in *four* planes" and "lines through *four* points," but these numbers I have not yet been able to determine.

5. It is useful also to examine the nature of the section of a developable made by a plane passing through a line of the system, and the nature of the cone whose vertex is on a line of the system. For these we shall find

$$\mu = r - 1, \quad \nu = n; \qquad \mu = m, \qquad \nu = r - 1;$$
$$\iota = \alpha + 1, \quad \kappa = m - 2; \qquad \iota = n - 2, \qquad \kappa = \beta + 1;$$
$$\tau = g - 1, \quad \delta = x - r + 4; \qquad \tau = y - r + 4, \quad \delta = h - 1.$$

These numbers being determined by reasoning precisely similar to that employed in § 3, little further explanation is necessary. I shall only observe in the case of the section that we have one point of inflection in addition to those arising from the stationary planes of the system, because one

of the "lines in two planes" coincides with the given line of the system, and gives rise, not to a double tangent, but to a tangent at a point of inflection. In like manner, for the cone whose vertex is on a line of the system, this line is a cuspidal edge of this cone in addition to those arising from the stationary points of the system.

We have an illustration of this in the cone drawn from a given point to touch a given surface. This has (*Journal*, vol. II. p. 69) $m(m-1)(m-2)$ cuspidal edges, but the curve of contact has in general no stationary points, and the cuspidal edges of this cone passing through the curve of contact, merely arise from the fact that $m(m-1)(m-2)$ lines of that system intersect in the vertex of that cone.

6. There is no difficulty in similarly determining the nature of a section made by a stationary plane or by a plane through two or more lines; nor in solving the corresponding questions for the cone. Before leaving this part of the subject, I wish to indicate the principal questions which remain to be solved, in order to complete that part of the theory of developables which involves no consideration of ordinary surfaces. I have already mentioned the determination of the number of "lines in four planes" and of "lines through four points." Moreover we have seen (§ 3) that every line of the system is intersected in general by $r-4$ other lines of the system. Therefore when r is greater than 4, the developable surface has in general beside its edge of regression another double line, being the locus of all the "points in two lines" of the system. It is required to determine the number of points where this nodal line meets the edge of regression. Also this nodal curve may be considered as a new system whose m indeed $= x$, but all whose other peculiarities remain to be determined. The lines of the new system are the intersections of the two planes of the first system corresponding to every two intersecting lines of it.

So likewise it is required to find the number of planes of a system which are also "planes through two lines," and to determine the peculiarities of the new system formed by the "planes through two lines" of a given system.

The lines of this new system are the lines joining the points of each pair of intersecting lines of the given system.

7. To this branch of the subject also belongs the ascertaining the number of conditions necessary to determine a given system. What has been already said enables us at least to fix a major limit to the number of conditions. Let

us suppose that we were given κ independent points, and required through these to determine a curve of the m^{th} degree. We have seen (§ 4) that the cone containing the curve whose vertex is any of these points, is of the $m-1^{\text{st}}$ degree. But we are given $\kappa - 1$ edges of this cone, namely, the lines joining the assumed point to all the rest. It follows therefore that $\kappa - 1$ cannot be greater than the number of conditions necessary to determine a cone of the $m-1^{\text{st}}$ degree; and hence that κ cannot be greater than $\dfrac{m(m+1)}{1.2}$. Thus for example we cannot describe a curve of the third degree through seven arbitrarily assumed points, because the six lines joining any one of these points to the rest will not in general lie on a cone of the second degree.

We might fix the limit for κ still lower if we took into consideration the multiple points which we have seen (§ 4) that the cone of the $m-1^{\text{st}}$ degree must possess; and should find that κ cannot exceed $\dfrac{(m+4)(m-1)}{2} - 2\beta - h$; or as we may otherwise express it; $2(r-1) + \dfrac{m-n}{2}$, or rather, since the same number of conditions determine a curve and its reciprocal, $2(r-1) - \left(\dfrac{m \sim n}{2}\right)$.

We shall return presently to this question, and shall only observe now, that being given a point on a curve of double curvature is algebraically equivalent to two conditions, since we are furnished with one condition from substituting the coordinates of the point in each of the equations of the curve: and also, that if we shall find that κ points determine a curve of the m^{th} degree, they do not necessarily determine it by a simple equation, but that we may on the contrary have several curves of the m^{th} degree passing through the κ given points.

8. I come now to point out an important difference in the causes which may give rise to double edges in the cone standing on a given curve of double curvature. It has been shewn that their number is h, the number of "lines through two points" which can be drawn through the vertex of the cone. But the two points need not necessarily be distinct, and if the curve have a double point, the line joining this point to the vertex of the cone will be a double edge of that cone. To an eye placed at any point, two different branches of the curve will appear to intersect if any line drawn through

the eye meet both branches; or in other words, if both branches be referred by projection to any plane, the intersections of their projections will be apparent points of intersection of the two branches; such points are the ordinary causes of double edges on the cone containing the curve whose vertex is at the point of view. Now if two branches of the curve should actually intersect, this point of intersection must of course be also a point of apparent intersection whatever be the point of view. As the phrase "lines through two points" has been used to include both these cases; when it is necessary to distinguish them, I shall say that a curve has so many *apparent* and so many *actual* double points, and the sum of the apparent and actual double points will be equal to h. A curve of double curvature, like a plane curve, has two tangent lines at every actual double point, namely the two lines of the system which correspond to each of the branches intersecting at that point. The plane of these two lines will be a double plane of the system corresponding to the double point. If the two tangent lines should coincide, the double point becomes a cusp or stationary point, noticed p. 19.

So in like manner for the section of the developable. When a curve has a double point, the double plane at that point gives rise to a double tangent in any section, in addition to those which arise from the "lines in two distinct planes." The consideration of the intersection of surfaces will throw further light on the subject of these double points.

9. Curves are connected with surfaces by the help of the principle that a curve of the m^{th} degree meets a surface of the κ^{th} degree in $m\kappa$ points.* This is true by our definitions when the surface of the κ^{th} degree consists of κ planes, and is therefore by the law of continuity true in general. Hence if more than $m\kappa$ points of a proper curve of the m^{th} degree lie in a surface of the κ^{th} degree, the whole curve must lie in that surface. We say a *proper* curve of the m^{th} degree, because if $p + q = m$, two curves of the p^{th} and q^{th} degrees might be considered as making up a curve of the m^{th} degree; and if *either* of these curves were wholly contained in the surface of the κ^{th} degree, we should have more than $m\kappa$ points common to the complex curve and to the surface.

Now since it is possible to describe a surface of the κ^{th}

* In like manner a developable of the n^{th} class has $n\kappa$ tangent planes common with a surface of the κ^{th} class.

degree through $\frac{(\kappa+1)(\kappa+2)(\kappa+3)}{1 \quad 2 \quad 3} - 1$ points, it is possible to describe a surface of the κ^{th} degree wholly containing a given proper curve of the m^{th} degree, provided that κ be such as to make $\frac{(\kappa+1)(\kappa+2)(\kappa+3)}{1 \quad 2 \quad 3} - 1 > m\kappa$. If κ be such as to make $\frac{(\kappa+1)(\kappa+2)(\kappa+3)}{1 \quad 2 \quad 3} - 1 > m\kappa + 1$, it would be possible to describe two distinct surfaces of the κ^{th} degree passing through the given curve, which may therefore be considered as the complete or partial intersection of two surfaces of the κ^{th} degree.

Thus, for example, every line of the 1^{st} degree must be a right line; for through two assumed points of it and any arbitrary point, we may describe a plane which must wholly contain the line of the first degree, since otherwise the line would cut the plane in two points: in like manner we can shew that a different plane must wholly contain the given line, which must therefore be the intersection of two planes.

So again, every proper curve of the second degree must be a plane curve. For through three points of it we may describe a plane which by the principles laid down must wholly contain the given curve.

The reader will have no difficulty in constructing by the same principles the following table, or in continuing it as far as he pleases.

Every curve of the m^{th} degree must be the partial intersection of two surfaces of the κ and l^{th} degrees,

$$m = 1; \quad \kappa = 1; \quad l = 1;$$
$$m = 2; \quad \kappa = 1; \quad l = 2;$$
$$m = 3; \quad \kappa = 2; \quad l = 2;$$
$$m = 4; \quad \kappa = 2; \quad l = 3;$$
$$m = 5; \quad \kappa = 3; \quad l = 3;$$
$$m = 6; \quad \kappa = 3; \quad l = 4;$$
&c.

This table however requires to be modified if we mean the surfaces κ and l to be always proper surfaces of those orders. For example, through a plane curve of the third order we can indeed describe an infinity of surfaces of the second order, but these will always be the plane of the curve and an arbitrary plane. Again, if a curve of the fifth degree lie altogether in a surface of the second degree, we may be able

to draw through it no other surface of the third degree, than the surface of the second degree and an arbitrary plane. We must therefore add to the foregoing table the plane curves of the different orders, and also the following possible cases:

$$m = 5; \quad \kappa = 2; \quad l = 4;$$
$$n = 6; \quad \kappa = 2; \quad l = 5;$$
&c.

We see hence that if we discuss all the cases which may occur of the intersection of two surfaces of the second degree, we shall be sure to have included all curves of the third degree; if we discuss the intersection of a surface of the second and of the third degrees, we shall include all curves of the fourth degree, and so on. We are led then to the problem to discuss the nature of the intersection of two surfaces of the m^{th} and n^{th} degrees.

10. The degree of the intersection of two such surfaces is mn, for any plane cuts the two surfaces in two curves which intersect in mn points.

At any of these points the intersection of the tangent planes to the two surfaces will be the tangent line to the curve of intersection. For any plane drawn through this line meets the surfaces in two curves which touch; such a plane therefore passes through two coincident points of the curve of intersection.

If however the two surfaces should touch, *every* plane through the point of contact meets the surfaces in two curves which touch. Every such plane therefore passes through two coincident points of the curve of intersection: we arrive then at the important result, that "if two surfaces touch, the point of contact is a double point on their curve of intersection."

We can prove that at such a point the curve of intersection has two tangents, by shewing that there are two directions in the common tangent plane, any plane through which meets the surfaces in curves which have three consecutive points in common. Take the tangent plane for the plane of xy, and let the equations of the surfaces be

$$z + Ax^2 + Bxy + Cy^2 + \&c. = 0,$$
$$z + A'x^2 + B'xy + C'y^2 + \&c. = 0.$$

Any plane $y = \mu x$ cuts the surface in curves which osculate, if we have $A + B\mu + C\mu^2 = A' + B'\mu + C'\mu^2$.

The two required directions are then given by the equation

$$(A - A')x^2 + (B - B')xy + (C - C')y^2 = 0.$$

11. Should the left-hand side of this equation be a perfect square, the two tangents at the double point coincide; the point of contact is therefore in this case a cusp or stationary point on the curve of intersection. We shall give the name of "stationary contact" to this species of contact, intermediate between simple contact and osculation.

We may illustrate this by the familiar example of the sphere described to touch a surface at the given point. The point of contact is in general an ordinary double point of the curve of intersection of the sphere and the surface; but if the centre of the sphere be the centre of either of the circles of greatest or least curvature, there is stationary contact, and the point of contact is a cusp on the curve of intersection.

When two surfaces osculate, every plane through the point of osculation meets the surfaces in curves having three consecutive points in common; the point of osculation is therefore a triple point on the curve of intersection, and we can shew as before that at such a triple point there may be drawn three tangents to the curve of intersection.

12. The number of apparent double points in the curve of intersection of two surfaces of the m^{th} and n^{th} degree $= \dfrac{mn(m-1)(n-1)}{1.2}$, (see Liouville, vol. x. p. 250, *Journal*, vol. II. p. 68).* As in the places cited the proof is not given of the algebraical principle on which this theorem depends, I shall add it here.

It is required to find the two conditions that two functions of a single variable of the m^{th} and n^{th} degree should have a quadratic factor in common. The only difficulty is to choose the two simplest from the different forms in which these conditions may be presented.

The method of elimination by symmetric functions gives us at once two conditions. Let the functions be ϕx, ψx, the roots of the first a, β, γ, &c., of the second a', β', γ'. Then if we form the equations,

$$\phi\beta'\,\phi\gamma'\,\phi\delta' + \phi a'\,\phi\gamma'\,\phi\delta' + \phi a'\,\phi\beta'\,\phi\delta'\ \&c. = 0,$$

$$\psi\beta\,\psi\gamma\,\psi\delta + \psi a\,\psi\gamma\,\psi\delta + \psi a\,\psi\beta\,\psi\delta\ \&c. = 0.$$

It is evident that these must be satisfied if any two of the quantities $\phi a'$, $\phi\beta'$, $\phi\gamma'$, &c., and of ψa, $\psi\beta$, $\psi\gamma$, &c. become $= 0$. If ϕ and ψ be functions of three variables (no

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* I was not aware, when I gave this result in the *Journal* that I had been anticipated by Mr. Cayley.

term of ϕ being of a higher degree in x, y, z, than m, nor of ψ than n), the first of these conditions is of the $m(n-1)$ in y, z, and the second of the $n(m-1)$; the two combined are satisfied for $mn(m-1)(n-1)$ points; which was to be proved.

13. Neither of these conditions, however, being symmetrical, it is natural to inquire whether we could not combine with the result of elimination between the equations, which is of the degree mn, another condition of the degree $(m-1)(n-1)$. This condition is found as follows: If we multiply ϕ by the $n-2$ factors of ψ not contained in ϕ, we must have the same result as when we multiply ψ by the $m-2$ factors of ϕ. If therefore we multiply ϕ by an arbitrary function of the $(n-2)$ degree, and equate the result to ψ multiplied by an arbitrary function of the $(m-2)$ degree; we have $m+n-2$ equations from which, eliminating the $m+n-4$ introduced indeterminate quantities, we have the two required conditions. Now suppose we set aside the equation obtained by equating the parts of the two products independent of x, the remaining equations differ only by the addition of terms from those which we should obtain if we had applied Euler's process to eliminate between the two equations deprived of their absolute terms: the results therefore, differing only by the addition of terms, will be of the same degree $(m-1)(n-1)$.

14. We have now sufficient data to determine the character of the curve of intersection of two surfaces of the degrees μ and ν. If the surfaces do not touch, we have

$$m = \mu\nu,$$

$$h = \frac{\mu\nu(\mu-1)(\nu-1)}{2},$$

$$\beta = 0;$$

and hence (see p. 22)

$$n = 3\mu\nu(\mu+\nu-3), \quad g = \frac{\mu\nu\left[\mu\nu\{3(\mu+\nu)-9\}^2 - 22(\mu+\nu)+71\right]}{2},$$

$$x = \frac{\mu\nu\{\mu\nu(\mu+\nu-2)^2 - 4(\mu+\nu)+8\}}{2},$$

$$r = \mu\nu(\mu+\nu-2), \quad \alpha = \mu\nu\{6(\mu+\nu)-20\},$$

$$y = \frac{\mu\nu\{\mu\nu(\mu+\nu-2)^2 - 10(\mu+\nu)+28\}}{2},$$

15. If the surfaces touch each other in one or more points, the number of apparent double points in the curve

of intersection is not affected, and the result is to add to the number of double edges on the cone standing on the curve of intersection. For there is an apparent double point whenever on any line through the point of view *two* values of the radius vector are the same for both surfaces: but a point of contact not being in general a double point on either surface, the radius vector to a point of contact has only *one* value the same for both surfaces; the point of contact is therefore not included in the number of apparent double points found by the methods alluded to in § 12.

Let the surfaces then have ordinary contact in t points and stationary contact in β points: the following are the characteristics of the curve of intersection. (The accentuated letters denote the values found in the preceding paragraph.)

$$m = m' \qquad g = g' + 22 + 28\beta + (6t + 8\beta)(3t + 4\beta - n');$$

$$a = a' - 12t - 15\beta; \quad x = x' + 4t + \tfrac{11}{2}\beta + \frac{(2t+3\beta)(2t+3\beta-2r')}{2};$$

$$n = n' - 6t - 8\beta;$$

$$r = r' - 2t - 3\beta; \quad h = h' + t;$$

$$\beta = \beta; \qquad y = y' + 10t + \tfrac{27}{2}\beta + \frac{(2t+3\beta)(2t+3\beta-2r')}{2}.$$

This is sufficient to complete the theory of the nature of the curve of intersection of two surfaces for all cases where that intersection does not break up into two curves of lower dimensions.

16. The foregoing also furnishes us a limit to the number of points at which two surfaces can touch, if their curve of intersection do not break up into lower curves. For a proper cone of the $\mu\nu^{\text{th}}$ degree cannot have more than $\dfrac{(\mu\nu-1)(\mu\nu-2)}{1.2}$ double edges; and since the cone of intersection has arising from apparent double points $\dfrac{\mu\nu(\mu-1)(\nu-1)}{1.2}$ double edges, those arising from contacts cannot exceed $\dfrac{\mu\nu(\mu+\nu-4)}{2} + 1$.

17. If the curve of intersection break up into two of the degrees p and q, the total number of apparent double points in the whole complex curve is the same as before; but this is made up of the sum of the apparent double points of the curves p and q, together with the number of apparent intersections of these curves. Now the pq lines in which the

cones standing on p and q intersect must correspond either to real or apparent intersections of these curves. If therefore we knew the number of *actual* intersections of these curves, by subtracting this number from pq we should know the number of apparent. At every point of *actual* intersection the complex curve has a double point, and the two surfaces therefore touch. Let us suppose then that we know that a given curve of the p^{th} degree is the partial intersection of two surfaces of the μ^{th} and ν^{th} degree, and that we are required to find the nature of the curve of the $\mu\nu - p$ degree, which is their remaining intersection. Let us also suppose that we have solved the problem 'to find at how many points of the curve p the two surfaces touch.' If these points of contact be not double points on the curve p, they must be points of intersection of p with the curve $\mu\nu - p$. Subtracting the number of such intersections from $p(\mu\nu - p)$ we have the number of their apparent intersections; and then knowing the number of apparent double points of the curve p, we can find those of the curve $(\mu\nu - p)$.

18. We proceed therefore to the solution of the question to which we were led in the last paragraph, "To find the number of points at which two surfaces touch along a curve which is not their complete intersection."

Let us first suppose this curve to be itself the complete intersection of two surfaces of the k^{th} and l^{th} degrees; so that the equations of the two given surfaces admit of being put in the form

$$AU + BV = 0,$$
$$CU + DV = 0,$$

where U, V, A, B, C, D are respectively functions of the degrees $k, l, \mu - k, \mu - l, \nu - k, \nu - l$.

Let A', B', C', D' be what A, B, C, D become when the coordinates of any point of UV are substituted in them, and let u, v be the tangent planes to U, V at the same point; then the tangent planes to the surfaces will be

$$A'u + B'v = 0,$$
$$C'u + D'v = 0.$$

The surfaces therefore will touch if $A'D' = B'C'$, and must therefore touch at the $kl(\mu + \nu - k - l)$ points, where the curve UV meets the surface $AD - BC$.

Mr. Cayley has suggested that by expressing this result in terms of the apparent double points of the curve UV, we shall have it in a form which will be probably true, even

when UV is not the complete intersection of two surfaces. If h be the number of apparent double points, we have

$$2h = kl(k-1)(l-1),$$

or

$$kl(k+l) = k^2l^2 + kl - 2h.$$

Making this change in the preceding formula, and writing p for kl, we learn that "if two surfaces of the orders μ, ν partially intersect in a curve of the degree p, having h apparent double points, and no actual double points or stationary points, they will touch along that curve in

$$p(\mu + \nu - p - 1) + 2h \text{ points.''}$$

19. This formula has as yet been proved only for the case where p is the complete intersection of two surfaces. We may see however, by reasoning analogous to that employed in the Calculus of Differences, that in the general case the formula expressing the number of points of contact must be of the form $p(\mu+\nu) + C$ (C being a constant depending on the form of the curve and not on the degree of the surfaces). For the effect of adding s to the degree of either surface is to add ps to the number of points of contact. This appears from the case where the first surface is a complex one, consisting of two of the degrees μ and s. We have then to add to the number of points of contact the ps points where the curve p meets the surface s, which are double points on the first surface.

We can now prove that the value of this constant is that given by the formula of the preceding paragraph whenever the curve p together with any other curve q, for which the formula is true, make up the complete intersection of two surfaces of the k^{th} and l^{th} degrees. That is to say, that if the intersection of the curves p and q can be represented by the formula

$$I = q(k + l - q - 1) + 2h',$$

it may also be represented by the formula

$$I = p(k + l - p - 1) + 2h.$$

For if we add these equations together, remembering that $p + q = kl$, we arrive at an equation already known to be true (§ 17), viz.

$$2h + 2h' + 2(pq - I) = kl(k-1)(l-1).$$

The second equation is therefore a consequence of the first.

This proof of the formula of the last paragraph is sufficient for all the cases to which we shall have occasion to apply it.

Equating the values just found for I, we are enabled, from knowing the number of apparent double points on the curve p, to find those of the curve q. For we get

$$2(h - h') = (k - 1)(l - 1)(p - q).$$

20. We have in the preceding paragraphs supposed that the curve p has no actual double points or stationary points. Let us suppose, however, that in the investigation in § 18 the surfaces UV touch each other in one or more points. Now it is plain that the equations

$$A'u + B'v = 0,$$
$$C'u + D'v = 0,$$

represent the same plane either when $\frac{A'}{B'} = \frac{C'}{D'}$, or when u and v represent the same plane. The points where $\frac{A'}{B'} = \frac{C'}{D'}$ are those where the curve p is met by the remaining part of the intersection of the surfaces μ, ν; the investigation of the number of such points is precisely the same as before; or "the number of points where the curve p is met by q depends solely on the number of its apparent double points, and not on that of its actual double points." If however we wish to know the whole number of points where the surfaces $\mu\nu$ touch along the curve UV, we must add to the number of points of intersection that of the number of actual double and stationary points.

The solution then of the problem proposed in § 17 is contained in the following theorem: "If the intersection of two surfaces of the orders μ, ν break up into two curves of the degrees p and q, the apparent double points of these curves are connected by the relation

$$2(\mathrm{a} - \mathrm{a}') = (\mu - 1)(\nu - 1)(p - q)."$$

21. This formula is true even if one or both the curves p, q be complex curves made up of others of lower dimensions: we must in this case consider the number of apparent double points in p, for example, as made up of the sum of the apparent double points of its component curves together with the apparent intersections of those curves.

Thus, for instance, we can verify the preceding formulæ for the case where the curve p is itself a complex curve consisting of two curves of the degrees π, π'; having y, y'

apparent double points, and intersecting each other in ρ points. We have then

$$\pi + \pi' = p,$$

and $$y + y' + (\pi\pi' - \rho) = \mathrm{a}.$$

Now π meets the remaining intersection in

$$\pi(\mu + \nu - \pi - 1) + 2y \text{ points};$$

but since ρ of these are on π', π meets q in

$$\pi(\mu + \nu - \pi - 1) + 2y - \rho \text{ points},$$

π' meets q in $\quad \pi'(\mu + \nu - \pi - 1) + 2y' - \rho.$

Adding these, we find for the number of points where p meets q, $\quad p(\mu + \nu - p - 1) + 2\mathrm{a}.$

22. We are now in possession of the materials requisite for the examination of the intersection of particular surfaces, and shall commence with the case where both surfaces are of the second degree. The table at § 9 shews that such an examination will include all possible curves of the third degree. We shall denote the number of apparent double points by a, and we have seen that the only species of lines of the first degree is the right line for which $\mathrm{a} = 0$: That the only species of proper curves of the second degree is the plane conic for which $\mathrm{a} = 0$: If the plane conic break up into two right lines in the same plane, we have still $\mathrm{a} = 0$, but $h = 1$: And if the line of the second degree break up into two right lines in different planes, we have $\mathrm{a} = 1$.

23. The curve of intersection of two surfaces of the second degree which do not touch has (§ 14) the following characteristics:

IV. 1. $\quad \mathrm{a} = 2;\ m = 4,\ n = 12,\ r = 8;\ g = 38,\ h = 2;$

$$\alpha = 16,\ \beta = 0;\ x = 16,\ y = 8.$$

If the surfaces touch at a single point, we have (§ 15)

IV. 2. $\quad \mathrm{a} = 2;\ m = 4,\ n = 6,\ r = 6;\ g = 6,\ h = 3;$

$$\alpha = 4,\ \beta = 0;\ x = 6,\ y = 4.$$

If at this point they have stationary contact, we have

IV. 3. $\quad \mathrm{a} = 2;\ m = 4,\ n = 4,\ r = 5;\ g = 2,\ h = 2;$

$$\alpha = 1,\ \beta = 1;\ x = 2,\ y = 2.$$

These are (by § 16) the only species of curves of the 4th degree which can arise from the intersection of two surfaces of the second degree.

If the surfaces touch in two points on the same generatrix of either surface, this right line must be common to both surfaces, and their curve of intersection breaks up into a line of the first and one of the third degree. The a of the latter is found by the formula of § 20, which, when we have $\mu = \nu = 2$, becomes $2(\mathrm{a} - \mathrm{a}') = p - q$, and therefore if $p = 1$, $q = 3$, $\mathrm{a} = 0$, we have $\mathrm{a}' = 1$. The following are therefore the characteristics of the proper nonplane curve of the third degree:

III. $\mathrm{a}=1,\ m=3,\ n=3,\ r=4;\ g=1,\ h=1,\ \alpha=0,\ \beta=0,\ x=0,\ y=0.$

If the two points of contact be not on the same generatrix, the curve of contact breaks up into two plane curves, and the points of contact are the points where the intersection of the planes meets the surface.

24. The properties of such curves of the third order are so well known that they may be briefly summed up here. Every such curve on a surface of the second degree is met twice by all the generatrices of one system, and once by those of the other. Every such curve is the intersection of two cones of the second degree, the vertices being any points on the curve. It is determined when six points of the curve are given (§ 7). Any plane of the system may be represented by the equation

$$At^3 + 3Bt^2 + 3Ct + D = 0,$$

where $ABCD$ represent planes and t is a variable parameter; and the corresponding developable is then given by the equation

$$(AD - BC)^2 = 4(B^2 - AC)(C^2 - BD),$$

the curve itself being the intersection of the cones $B^2 - AC$, $C^2 - BD$, which have the edge BC common.

Beside the nonplane curve of the third degree there are of course plane curves of the third degree, for which $\mathrm{a} = 0$; and also the following improper systems of the third degree.

$\mathrm{a} = 1$. A plane line of the second degree and a right line once meeting the line of the second degree.

$\mathrm{a} = 2$. A plane line of the second degree and a right line not meeting it.

$\mathrm{a} = 3$. Three right lines, no two of which are in the same plane.

25. We proceed now to discuss in like manner the intersection of a surface of the second with one of the third degree.

If the surfaces do not touch, we have

VI. 1. $\mathrm{a} = 6$; $m = 6$, $n = 36$, $r = 18$; $g = 531$, $h = 6$;
$$\alpha = 60,\ \beta = 0;\ x = 126,\ y = 96.$$

The surfaces however may touch at 1, 2, 3 or 4 points, and at any or all of these there may be stationary contact. This gives rise to 14 other varieties of curves of the sixth order whose characteristics can be obtained without difficulty. We give as an example the last species,

VI. 15. $\mathrm{a} = 6$; $m = 6$, $n = 4$, $r = 6$; $g = 3$, $h = 6$;
$$\alpha = 0,\ \beta = 4;\ x = 4,\ y = 6.$$

Next let the curve break up into a right line and a line of the fifth degree. The formula of § 20 becomes, when $\mu = 3$, $\nu = 2$,
$$\mathrm{a} - \mathrm{a}' = p - q.$$

In the present case, therefore, we see that the curve of the fifth degree has $\mathrm{a} = 4$; and since a cone of the fifth degree may have six double edges, it appears that the surfaces may touch at two points of the curve of the fifth degree. This gives rise to the six following varieties:

V. 1. $\mathrm{a} = 4$; $m = 5$, $n = 21$, $r = 12$; $g = 156$, $h = 4$;
$$\alpha = 32,\ \beta = 0;\ x = 48,\ y = 32.$$

V. 2. $\mathrm{a} = 4$; $m = 5$, $n = 15$, $r = 10$; $g = 70$, $h = 5$;
$$\alpha = 20,\ \beta = 0;\ x = 30,\ y = 20.$$

V. 3. $\mathrm{a} = 4$; $m = 5$, $n = 13$, $r = 9$; $g = 48$, $h = 4$;
$$\alpha = 17,\ \beta = 1;\ x = 22,\ y = 14.$$

V. 4. $\mathrm{a} = 4$; $m = 5$, $n = 9$, $r = 8$; $g = 20$, $h = 6$;
$$\alpha = 8,\ \beta = 0;\ x = 16,\ y = 12.$$

V. 5. $\mathrm{a} = 4$; $m = 5$, $n = 7$, $r = 7$; $g = 10$, $h = 5$;
$$\alpha = 5,\ \beta = 1;\ x = 10,\ y = 8.$$

V. 6. $\mathrm{a} = 4$; $m = 5$, $n = 5$, $r = 6$; $g = 4$, $h = 4$;
$$\alpha = 2,\ \beta = 2;\ x = 5,\ y = 5.$$

We see hence that when the intersection breaks up into lines of the first and fifth degrees, the surfaces cannot touch in more than 5 points: 3 common to both lines, 2 peculiar to the latter.

Next let the curve of intersection break up into lines of the second and fourth degrees. We shall obtain every possible curve of the fourth degree if we examine in turn each of the possible curves of the second degree which may unite with it to form the intersection of two surfaces of the second and third degrees.

If for the curve of the second degree $\mathrm{a} = 0$, for that of the fourth degree $\mathrm{a} = 2$. This only gives us the three species already discussed in § 22: and in fact it is easy to see directly, that if two surfaces of the second and third degrees have common a plane line of the second degree, their remaining intersection must lie also upon another surface of the second degree.

If however for the line of the second degree $\mathrm{a} = 1$, that is, if it consist of two right lines not in the same plane, for that of the fourth degree $\mathrm{a} = 3$. This is a curve of the fourth degree altogether distinct from those discussed in § 23. Its characteristics are

$$\text{IV. 4.}\quad \mathrm{a} = 3;\ m = 4,\ n = 6,\ r = 6;\ g = 6,\ h = 3;$$
$$\alpha = 4,\ \beta = 0;\ x = 6,\ y = 4.$$

This system is the reciprocal of the system VI. 15. It only differs from the system IV. 2, in that its "lines through two points" wholly arise from apparent double points, while in IV. 2, one is caused by an actual double point.

In this case the surfaces cannot touch in more than six points. We might shew directly that if the intersection break up into two right lines not in same plane and a proper curve of the fourth degree, the surfaces cannot touch at any other point. For if they did, the generatrix at this point which meets the two given lines must (contrary to hypothesis) lie altogether in the surface of the third degree, which otherwise it would meet in four points.

It is unnecessary to examine the case where the intersection breaks up into two systems of the third degree, since we know that this can give rise to no species of curve but those already examined. I shall only observe that in this case $\mathrm{a} = \mathrm{a}'$. Thus if the surfaces cut in 3 lines not in same plane ($\mathrm{a} = 3$), they must also meet in three other lines not in same plane. This may also be easily proved directly.

26. To the four species just enumerated of nonplane curves of the fourth degree, we must add the plane curves of the fourth degree $\mathrm{a} = 0$, and also the following improper lines of the fourth degree.

a = 2. 1. A plane system of third degree and a right line once meeting it.
2. A system of third degree for which a = 1 and a right line twice meeting it.
3. Two plane systems of second degree intersecting each other in two points.

a = 3. 1. A plane system of third degree and a right line not meeting it.
2. A system of third degree for which a = 1, and a right line once meeting it.
3. Two plane systems of second degree intersecting in one point.

a = 4. 1. A system of third degree for which a = 1, and a right line not meeting it.
2. Two plane nonintersecting systems of the second.

a = 5. A plane line of second degree and two right lines meeting neither it nor each other.

a = 6. Four right lines, no two of which are in the same plane.

26. Knowing the different possible systems of the fourth degree, we can obtain those of the fifth. First, if the curve of the fifth degree be the partial intersection of two of the third, their remaining intersection is a curve of the fourth degree. The equation of § 20, when $\mu = \nu = 3$, becomes $a - a' = 2(p - q)$, and, applied to this case, gives $a - a' = 2$. If therefore for the curve of the fourth degree a = 2, for that of the fifth a = 4. This gives us the class of curves already discussed, and in fact we can see that if two surfaces,

$$aU + bV = 0,$$
$$cU + dV = 0,$$

pass through the intersection of two surfaces of the second degree, their remaining intersection lies on a surface of the second degree
$$ad - bc = 0.$$

If for the system of the fourth degree a = 3, for that of the fifth a = 5. It appears that this curve of the fifth degree may have one actual double point; this gives rise to the three following varieties:

v. 7. a = 5; $m = 5$, $n = 5$, $\beta = 0$, as in v. 2.

v. 8. a = 5; $m = 5$, $h = 6$, $\beta = 0$, the rest as in v. 4.

v. 9. a = 5; $m = 5$, $h = 5$, $\beta = 1$, the rest as in v. 5.

This example shews that the intersection of two surfaces may consist of two curves, neither of which is the complete intersection of any two surfaces.

Lastly, if for the system of the fourth degree $a = 4$, for that of the fifth degree $a = 6$. This gives the system

v. 10. $a = 6$; $m = 5$, $h = 6$, $\beta = 0$, &c., as in v. 4.

Secondly, let the curve of the fifth degree be the intersection of surfaces of the second and fourth degrees; the remaining intersection is of the third degree, and the equation for this case gives us $a - a' = 3$. We have hence systems of the fifth degree $a = 4$, the same as those already discussed; but also $a = 5$, $a = 6$, which, if no consideration of surfaces be introduced, do not differ from the systems v. 7, 8, 9, 10; but yet which lie in a surface of the second degree, and through which no proper surface of the third degree can pass. There is evidently no difficulty in enumerating in like manner the different possible species of curves of higher degrees.

27. We shall speak briefly of the properties of curves of the fourth order. The class $a = 2$ is evidently determined by 8 points, since through these we can draw an infinity of surfaces of the second order. Mr. Cayley has noticed that the cone of the third degree, whose vertex is any point of the curve, is also the locus of the generatrices, at that point, of the several surfaces of the second degree passing through that point. For let the equations of any two such surfaces be

$$u + x = 0,$$
$$v + y = 0,$$

x and y being the tangent planes, then the cone through their intersection is $uy - vx = 0$, passing through ux and vy; or in other words, it is easy to see independently that every generatrix of each of the surfaces is a "line through two points" of the curve. I have already given a construction for "the lines through two points" at any assumed point (*Journal*, vol. II. p. 68). We may now, if we please, substitute the following: "Through the assumed point and the 8 determining the curve, draw a surface of the second degree, its generatrices through the assumed point will be the lines required."

28. The class $a = 3$ lie on a surface of the second degree. Each generatrix of one system meets the curve once, and each generatrix of the other system three times. For any generatrix meets three times any surface of the third degree

through the curve. But such a surface meets that of the second in the curve and in two generatrices of the same system: if therefore the assumed generatrix be of that system, since it does not meet the right lines at all, it must meet the curve thrice; if of the other system, since it meets the right lines twice, it only meets the curve once. We learn therefore (§ 4) that the cone whose vertex is any point of the curve has a double line; namely, one of the generatrices at the point, or that "Every such curve may be regarded as the intersection of a surface of the second degree with a cone of the third, having one of the generatrices of that surface for a double edge."*

This curve can obviously have no "lines through *four* points." It is the reciprocal of the system VI. 15 expressed by the equation $At^4 + 4Bt^3 + 6Ct^2 + 4Dt + E = 0$, already discussed (*Crelle*, vol. XXXIV. p. 15; *Journal*, vol. III. p. 69). The locus of the "points on two lines" of this latter system is a system IV. 4, intersecting the system VI. 15 at its four stationary points.

29. Mr. Cayley has proved by the following reasoning *à priori*, that every curve of the fourth degree is determined by 8 points. Take any "plane through two lines" of the system; the two cones whose vertices are the points corresponding to the two lines are touched by that plane; this intersection therefore consists of the line joining the points *twice;* the given curve of the fourth degree and a certain curve of the *third* degree. Conversely, any curve of the fourth degree may be obtained from a curve of the third degree as follows: Take two points on any line which meets the curve once; the cones whose vertices are these points have a common tangent plane, namely, the plane of the line

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* We had not included in our enumeration (§ 9) the case where the surfaces have in common a right line which is a double line on the surface of the third degree. We are led to infer now that this gives rise to the same kind of curve as when they have in common two right lines not in the same plane. It would lengthen this paper too much to give the complete examination of the case where part of the intersection of two surfaces is a double line on one of them. In this case the reader will perceive from § 15 that there will be points on the double line which will be counted among the apparent double points; and a process similar to that already employed leads to the following formula: "If two surfaces of the orders μ, ν, have common a curve of the degree p, having a apparent double points, which curve is a double line on the surface μ, their remaining intersection will be such that

$$a' - 4a = \frac{(p' - 2p)(\mu - 1) + 2p}{2}(\nu - 1)."$$

and of the line of the system at the point where the curve is met by the line; the two cones therefore intersect in a curve of the fourth degree. Now a curve of the third degree is determined by 6 points or 12 constants; a right line meeting the curve gives three constants more; and the two arbitrary points to be assumed on that line two constants more, or 17 in all. He shews that this number is reduced by one, because the same curve of the fourth degree can be obtained from different curves of the third degree, by properly assuming the intersecting lines and vertices. The curve is therefore determined by 16 constants or 8 points.

Mr. Cayley gives the following construction for describing a curve IV. 4 through 8 points. Take any one of them as the vertex of a cone; the locus of the double edges of all cones having a double edge, and passing through the remaining seven points, will be a certain cone; and we can then draw a surface of the second degree through the 8 points and having a generatrix coincident with some side of this cone. The intersection of this surface with the cone whose double edge is a generatrix of that surface will be the curve of the fourth order required.

30. Let us inquire of how many solutions Mr. Cayley's construction admits. Let three of the cones whose vertex is at the point x, y, z, and which pass through seven other points, be s, s', s''; then the double edges of all such cones will lie on the cone

$$\frac{ds}{dx}\left(\frac{ds'}{dy}\frac{ds''}{dz}-\frac{ds'}{dz}\frac{ds''}{dy}\right)+\frac{ds}{dy}\left(\frac{ds'}{dz}\frac{ds''}{dx}-\frac{ds'}{dx}\frac{ds''}{dy}\right)$$
$$+\frac{ds}{dz}\left(\frac{ds'}{dx}\frac{ds''}{dy}-\frac{ds'}{dy}\frac{ds''}{dx}\right)=0.$$

This is a cone of the sixth degree, and of which the seven given edges are double edges.

Now the problem "to describe a surface of the second degree through eight points and having a generatrix on a cone of the n^{th} degree whose vertex is at one of the points" admits in general of $3n$ solutions. The present problem appears therefore to admit of 18 solutions. Fourteen of these however are irrelevant, since they relate to the seven cones which can be described having one of the given edges for a double edge. Through eight points therefore we can describe one curve of the class IV. 1, and four of the class IV. 4.

31. Curves of the fifth degree, for which a = 4, lie on a surface of the second degree, and are met thrice by the generatrices of one system, and twice by those of the other. The cone whose vertex is any point of the curve has one generatrix for a double edge and the other for a single edge, and consequently meets the surface of the second degree nowhere else save on the curve. Mr. Cayley shews as follows, that such a curve is determined by 11 points on the surface of the second degree, or by 20 conditions. For being given eleven such points, we are given the surface of the second degree, and if through the 11 points, through 4 points on a generating line, and through 4 points not on the surface we describe a surface of the third degree, this must be a proper surface, since the last 4 points are supposed not to be in the same plane; it must contain the assumed generating line, which would otherwise meet the surface in 4 points, and it therefore meets the surface of the second degree in a curve of the fifth degree. Two curves of the fifth degree can be described according as the assumed generatrix is of one or the other system; but all generatrices of the same system give rise to the same curve of the fifth degree.

With regard to the class a = 5, I have nothing satisfactory to offer: but the class a = 6 is the reciprocal of the system

$$At^5 + 5Bt^4 + 10Ct^3 + 10Dt^2 + 5Et + F.$$

I have found the equation of the resulting developable, but I do not give it here, as I have not got it in a form to exhibit the cuspidal and nodal edges of that surface. This system is determined by 10 points or 20 constants, and in general the system $At^k + Bt^{k-1} + Ct^{k-2} + \&c.$ is determined by $4k$ conditions. For the equation involves $m+1$ planes, and has therefore $4m+3$ arbitrary constants. But the equation in its most general form must contain 3 indeterminate constants, for it is not necessary to determine which plane of the system answers to $t = 0$ or $t = \infty$; and the system remains the same if we substitute for t, $\dfrac{A(t-t')}{t-t''}$. Thus the system $(Ax+By)+t.(A'x+B'y)$ is precisely the same system as $x+ty$. There remain therefore but $4k$ conditions to be used in determining the system. I have not however been able to ascertain the number of conditions in general necessary to determine a given curve.

The characteristics of the system

$$At^k + Bt^{k-1} + \&c.,$$

are

$$m = 3(k-2),\ n = k,\ r = 2(k-1),\ \alpha = 0,\ \beta = 4(k-3),$$

$$g = \frac{(k-1)(k-2)}{2},\ h = \frac{9k^2 - 53k + 80}{2},\ x = 2(k-2)(k-3),$$

$$y = 2(k-1)(k-3).$$

July 21, 1849.

ON THE DEVELOPABLE SURFACES WHICH ARISE FROM TWO SURFACES OF THE SECOND ORDER.

By Arthur Cayley.

Any two surfaces considered in relation to each other give rise to a curve of intersection, or, as I shall term it, an Intersect and a circumscribed Developable* or Envelope. The Intersect is of course the edge of regression of a certain Developable which may be termed the Intersect-Developable, the Envelope has an edge of regression which may be termed the Envelope-Curve. The order of the Intersect is the product of the orders of the two surfaces, the class of the Envelope is the product of the classes of the two surfaces. When neither the Intersect breaks up into curves of lower order, nor the Envelope into Developables of lower class, the two surfaces are said to form a proper system. In the case of two surfaces of the second order (and class) the Intersect is of the fourth order and the Envelope of the fourth class. Every proper system of two surfaces of the second order belongs to one of the following three classes:—*A*. There is no contact between the surfaces; *B*. There is an ordinary contact; *C*. There is a singular contact. Or the three classes may be distinguished by reference to the conjugates (conjugate points or planes) of the system. *A*. The four conjugates are all distinct; *B*. Two conjugates coincide; *C*. Three conjugates coincide. To explain this it is necessary to remark that in the general case of two surfaces of the second order not in contact (*i.e.* for systems of the class *A*) there is

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* The term 'Developable' is used as a substantive, as the reciprocal to 'Curve,' which means curve of double curvature. The same remark applies to the use of the term in the compound Intersect-Developable. For the signification of the term 'singular contact,' employed lower down, see Mr. Salmon's memoir 'On the Classification of Curves of Double Curvature,' p. 23.

a certain tetrahedron such that with respect to either of the surfaces (or more generally with respect to any surface of the second order passing through the Intersect of the system or inscribed in the Envelope) the angles and faces of the tetrahedron are reciprocals of each other, each angle of its opposite face, and *vice versâ*. The angles of the tetrahedron are termed the conjugate points of the system, and the faces of the tetrahedron are termed the conjugate planes of the system, and the term conjugates may be used to denote indifferently either the conjugate planes or the conjugate points. A conjugate plane and the conjugate point reciprocal to it are said to correspond to each other. Each conjugate point is evidently the point of intersection of the three conjugate planes to which it does not correspond, and in like manner each conjugate plane is the plane through the three conjugate points to which it does not correspond. In the case of a system belonging to the class B, two conjugate points coincide together in the point of contact forming what may be termed a double conjugate point, and in like manner two conjugate planes coincide in the plane of contact (*i.e.* the tangent plane through the point of contact) forming what may be termed a double conjugate plane. The remaining conjugate points and planes may be distinguished as single conjugate points and single conjugate planes. It is clear that the double conjugate plane passes through the three conjugate points, and that the double conjugate point is the point of intersection of the three conjugate planes: moreover each single conjugate plane passes through the single conjugate point to which it does not correspond and the double conjugate point; and each single conjugate point lies on the line of intersection of the single conjugate plane to which it does not correspond and the double conjugate plane.

In the case of a system belonging to the class (C), three conjugate points coincide together in the point of contact forming what may be termed a triple conjugate point, and three conjugate planes coincide together in the plane of contact forming a triple conjugate plane. The remaining conjugate point and conjugate plane may be distinguished as the single conjugate point and single conjugate plane. The triple conjugate plane passes through the two conjugate points and the triple conjugate point lies on the line of intersection of the two conjugate planes, the single conjugate plane passes through the triple conjugate point and the single conjugate point lies on the triple conjugate plane.

Suppose now that it is required to find the Intersect-Developable of two surfaces of the second order. If the equations of the surfaces be $\Upsilon = 0$, $\Upsilon' = 0$ (Υ, Υ' being homogeneous functions of the second order of the coordinates ξ, η, ζ, ω), and x, y, z, w represent the coordinates of a point upon the required developable surface: if moreover U, U' are the same functions of x, y, z, w that Υ, Υ' are of ξ, η, ζ, ω and X, Y, Z, W; X', Y', Z', W' denote the differential coefficients of U, U' with respect to x, y, z, w; then it is easy to see that the equation of the Intersect-Developable is obtained by eliminating ξ, η, ζ, ω between the equations

$$\Upsilon = 0,\ \Upsilon' = 0,$$
$$X\xi + Y\eta + Z\zeta + W\omega = 0,$$
$$X'\xi + Y'\eta + Z'\zeta + W'\omega = 0.$$

If, for shortness, we suppose

$$\begin{aligned} \overline{F} &= YZ' - Y'Z, & \overline{L} &= XW' - X'W, \\ \overline{G} &= ZX' - Z'X, & \overline{M} &= YW' - Y'W, \\ \overline{H} &= XY' - X'Y, & \overline{N} &= ZW' - Z'W, \end{aligned}$$

(values which give rise to the identical equation

$$\overline{L}\,\overline{F} + \overline{M}\,\overline{G} + \overline{N}\,\overline{H} = 0),$$

then, λ, μ, ν, ρ denoting any indeterminate quantities, the two linear equations in ξ, η, ζ, ω are identically satisfied by assuming

$$\begin{aligned} \xi &= \quad . \quad \overline{N}\mu - \overline{M}\nu + \overline{F}\rho, \\ \eta &= -\overline{N}\lambda \quad . \quad + \overline{L}\nu + \overline{G}\rho, \\ \zeta &= \overline{M}\lambda - \overline{L}\mu \quad . \quad + \overline{H}\rho, \\ \omega &= -\overline{F}\lambda - \overline{G}\mu - \overline{H}\nu \quad . \end{aligned}$$

and, substituting these values in the equations $\Upsilon = 0$, $\Upsilon' = 0$, we have two equations:

$$A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu$$
$$+ 2L\lambda\rho + 2M\mu\rho + 2N\nu\rho = 0,$$
$$A'\lambda^2 + B'\mu^2 + C'\nu^2 + 2F'\mu\nu + 2G'\nu\lambda + 2H'\lambda\mu$$
$$+ 2L'\lambda\rho + 2M'\mu\rho + 2N'\nu\rho = 0,$$

which are of course such as to permit the four quantities λ, μ, ν, ρ to be simultaneously eliminated. The coefficients of these equations are obviously of the fourth order in x, y, z, w.

Suppose for a moment that these coefficients (instead of being such as to permit this simultaneous elimination of λ, μ, ν, ρ) denoted any arbitrary quantities, and suppose that the indeterminates λ, μ, ν, ρ were besides connected by two linear equations,

$$a\lambda + b\mu + c\nu + d\rho = 0,$$

$$a'\lambda + b'\mu + c'\nu + d'\rho = 0.$$

Then, putting

$$bc' - b'c = f, \quad ad' - a'd = l,$$

$$ca' - c'a = g, \quad bd' - b'd = m,$$

$$ab' - a'b = h, \quad cd' - c'd = n,$$

(values which give rise to the identical equation $lf + mg + nh = 0$), and effecting the elimination of λ, μ, ν, ρ between the four equations, we should obtain a final equation $\square = 0$, in which $\square$ is a homogeneous function of the second order in each of the systems of coefficients A, B, &c. and A', B', &c. and a homogeneous function of the fourth order (indeterminate to a certain extent in its form on account of the identical equation $lf + mg + nh = 0$) in the coefficients f, g, h, l, m, n.* But reestablishing the actual values of the coefficients A, B, &c., A', B', &c. (by which means the function $\square$ becomes a function of the sixteenth order in x, y, z, w) the quantities f, g, h, l, m, n ought, it is clear, to disappear of themselves; and the way this happens is that the function $\square$ resolves itself into the product of two factors M and Ψ, the latter of which is independent of f, g, h, l, m, n. The factor M is consequently a function of the fourth order in these quantities, and it is also of the eighth order in the variables x, y, z, w. The factor Ψ is consequently of the eighth order in x, y, z, w. And the result of the elimination being represented by the equation $\Psi = 0$, the Intersect-Developable in the general case, or what is the same thing for systems of the class (A), is of the eighth order. In the case of a system of the class (B)

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* I believe the result of the elimination is

$$\square = 4(PR - Q^2) = 0,$$

where if we write $uA + u'A' = A$, &c. the quantities P, Q, R are given by the equation (identical with respect to u, u')

$$Pu^2 + 2Quu' + Ru'^2 = (Aa^2 + ..)(Aa'^2 + ..) - (Aaa' + ..)$$

$$= u^2\{(BC - F^2)f^2 + ..\} + uu'\{(BC' + B'C - 2FF')f^2 + ..\} + u'^2\{(B'C' - F'^2)f^2 + ..\}$$

a theorem connected with that given in the second part of my memoir 'On Linear Transformations' (*Journal*, vol. I. p. 109). I am not in possession of any verification *a posteriori* of what is subsequently stated as to the resolution into factors of the function $\square$ and the forms of these factors.

the equation obtained as above contains as a factor the square, and in the case of a system of the class (C) the cube of the linear function, which equated to zero is the equation of the plane of contact. The Intersect-Developable of a system of the class (B) is therefore a Developable of the sixth order, and that of a system of the class (C) a Developable of the fifth order. The elimination is in every case most simply effected by supposing two of the quantities λ, μ, ν, ρ to vanish (*e.g.* ν and ρ): the equations between which the elimination has to be effected then are

$$A\lambda^2 + B\mu^2 + 2H\lambda\mu = 0,$$
$$A'\lambda^2 + B'\mu^2 + 2H'\lambda\mu = 0.$$

And the result may be presented under the equivalent forms

$$(AB' + A'B - 2HH')^2 - 4(AB - H^2)(A'B' - H'^2) = 0,$$
$$(AB' - A'B)^2 + 4(AH' - A'H)(BH' - B'H) = 0,$$

the latter of which is the most convenient. These two forms still contain an extraneous factor of the eighth order in x, y, z, w, of which they can only be divested by substituting the actual values of A, B, H, A', B', H'.

A. Two surfaces forming a system belonging to this class may be represented by equations of the form

$$ax^2 + by^2 + cz^2 + dw^2 = 0,$$
$$a'x^2 + b'y^2 + c'z^2 + d'w^2 = 0,$$

where $x = 0$, $y = 0$, $z = 0$, and $w = 0$, are the equations of the four conjugate planes. There is no particular difficulty in performing the operations indicated by the general process given above; and if we write, in order to abbreviate,

$$bc' - b'c = f, \quad ad' - a'd = l,$$
$$ca' - c'a = g, \quad bd' - b'd = m,$$
$$ab' - a'b = h, \quad cd' - c'd = n,$$

(values which satisfy the identical equation $lf + mg + nh = 0$), the result after all reductions is

$$\begin{aligned} l^2f^4y^4z^4 &+ m^2g^4z^4x^4 + n^2h^4x^4y^4 + l^4f^2x^4w^4 + m^4g^2y^4w^4 + n^4h^2z^4w^4 \\ &+ 2mng^2h^2x^4y^2z^2 + 2nlh^2f^2y^4z^2x^2 + 2lmf^2g^2z^4x^2y^2 \\ &- 2m^2n^2ghw^4y^2z^2 - 2n^2l^2hfw^4z^2x^2 - 2l^2m^2fgw^4x^2y^2 \\ &+ 2fmg^2l^2x^4z^2w^2 + 2gnh^2m^2y^4x^2w^2 + 2hlf^2n^2z^4y^2w^2 \\ &- 2fnh^2l^2x^4y^2w^2 - 2glf^2m^2y^4z^2w^2 - 2hmg^2n^2z^4x^2w^2 \\ &+ 2(mg - nh)(nh - lf)(lf - mg)\,x^2y^2z^2w^2 = 0, \end{aligned}$$

which is therefore the equation of the Intersect-Developable for this case. The discussion of the geometrical properties of the surface will be very much facilitated by presenting the equation under the following form, which is evidently one of a system of six different forms,

$$\{m(gx^2+nw^2)(hy^2-gz^2+lw^2)-l(-fy^2+nw^2)(-hx^2+fz^2+mw^2)\}^2$$
$$-\ 4fglmx^2y^2(hy^2-gz^2+lw^2)(-hx^2+fz^2+mw^2)=0.$$

B. Two surfaces forming a system belonging to this class may be represented by equations such as

$$ax^2+by^2+cz^2+2nzw=0,$$
$$a'x^2+b'y^2+c'z^2+2n'zw=0,$$

in which $x=0$, $y=0$ are the equations of the single conjugate planes, $z=0$ that of the double conjugate plane or plane of contact, $w=0$ that of an indeterminate plane through the two single conjugate points. If we write

$$bc'-b'c=f, \quad an'-a'n=p,$$
$$ca'-c'a=g, \quad bn'-b'n=q,$$
$$ab'-a'b=h, \quad cn'-c'n=r,$$

(values which satisfy the identical equation $pf+qg+rh=0$), the result after all reductions is

$$r^4h^2z^6+2pr^2h(rh-qg)z^4x^2-2qr^2h(pf-rh)z^4y^2+4p^2qr^2hz^3x^2w$$
$$-\ 4pq^2r^2hz^3y^2w+p^2(rh-qg)^2z^2x^4+q^2(pf-rh)^2z^2y^4$$
$$+\ 2pq(4r^2h^2-fgpq)z^2x^2y^2+4p^3q(rh-qg)zx^4w+4pq^3zy^4w$$
$$-\ 4p^2q^2(qg-pf)zx^2y^2w+4p^4q^2x^4w^2+4p^2q^4y^4w^2+8p^3q^3x^2y^2w^2$$
$$+\ 4p^2qrh^2x^4y^2+4pq^2rh^2x^2y^4=0,$$

which is therefore the equation of the Intersect-Developable for systems of the case in question. The equation may also be presented under the form

$$\{q(px^2+rz^2)(hy^2-gz^2+2pzw)-p(qy^2+rz^2)(-hx^2+fz^2+2qzw)\}^2$$
$$+\ 4p^2q^2x^2y^2(hy^2-gz^2+2pzw)(-hx^2+fz^2+2qzw)=0,$$

which it is to be remarked contains the extraneous factor z^2. The following is also a form of the same equation,

$$\{r(qg-pf)z^3-fp^2zx^2+gq^2zy^2+2pq(px^2+qy^2)w\}^2$$
$$-4pq(px^2+qy^2+rz^2)\{r(hy^2-gz^2)(fz^2-hx^2)+2pq(gx^2-fy^2)zw\}=0.$$

C. Two surfaces forming a system belonging to this class may be represented by equations of the form

$$ax^2 + by^2 + 2fyz + 2nzw = 0,$$
$$a'x^2 + b'y^2 + 2f'yz + 2n'zw = 0,$$

in which $bn' - b'n = 0$, $af' - a'f = 0$. In these equations $x = 0$ is the equation of a properly chosen plane passing through the two conjugate points, $y = 0$ is the equation of the single conjugate plane, $z = 0$ that of the triple conjugate plane, and $w = 0$ is the equation of a properly chosen plane passing through the single conjugate point. Or without loss of generality, we may write

$$\alpha\,(x^2 - 2yz) + \beta\,(y^2 - 2zw) = 0,$$
$$\alpha'(x^2 - 2yz) + \beta'(y^2 - 2zw) = 0,$$

where x, y, z and w have the same signification as before.* The result after all reductions is

$$4z^3w^2 + 12z^2x^2w + 9zx^4 - 24zxy^2w - 4x^3y^2 + 8y^4w = 0,$$

which may also be presented under the forms

$$z(x^2 - 2wz)^2 - 4x\,(x^2 - 2wz)\,(y^2 - 2zx) + 8w\,(y^2 - 2zx)^2 = 0,$$

and $\qquad z\,(3x^2 + 2wz)^2 - 4y^2\,(x^3 - 2wy^2 + 6zxw) = 0.$

Proceeding next to the problem of finding the envelope of two surfaces of the second order, this is most readily effected by the following method communicated to me by Mr. Salmon. Retaining the preceding notation, the equation $U + kU' = 0$ belongs to a surface of the second order passing through the

* Of course in working out the equation of the Intersect-Developable, it is simpler to employ the equations $x^2 - 2yz = 0$, $y^2 - 2zw = 0$. These equations belong to two cones which pass through the intersect and have their vertices in the triple conjugate point and single conjugate point respectively. I have not alluded to these cones in the text, as the theory of them does not come within the plan of the present memoir, the immediate object of which is to exhibit the equations of certain developable surfaces—but these cones are convenient in the present case as furnishing the easiest means of defining the planes $x = 0$, $w = 0$. If we represent for a moment the single conjugate point by S and the triple conjugate point by T (and the cones through these points by the same letters), then the point T is a point upon the cone S, and the triple conjugate plane which touches the cone S along the line TS touches the cone T along some generating line TM. Let the other tangent plane through the line TS to the cone T be TM', where M' may represent the point where the generating line in question meets the cone S; and we may consider M as the point of intersection of the line TM with the tangent plane through the line SM' to the cone S. Then the plane TMM' is the plane represented by the equation $x = 0$, and the plane SMM' is that represented by the equation $w = 0$. We may add that $y = 0$ is the equation of the plane TSM', and $z = 0$ that of the plane TSM.

Intersect of the two surfaces $U=0$, $U'=0$. The polar reciprocal of this surface $U+kU'=0$ is therefore a surface inscribed in the envelope of the reciprocals of the two surfaces $U=0$, $U'=0$, and consequently this envelope is the envelope (in the ordinary sense of the word) of the reciprocal of the surface $U+kU'=0$, k being considered as a variable parameter. It is easily seen that the reciprocal of the surface $U+kU'=0$ is given by an equation of the form

$$\mathrm{A} + 3\mathrm{B}k + 3\mathrm{C}k^2 + \mathrm{D}k^3 = 0,$$

in which A, B, C, D are homogeneous functions of the second order in the coordinates x, y, z, w. Differentiating with respect to k, and performing the elimination, we have for the equation of the envelope in question,

$$(\mathrm{AD}-\mathrm{BC})^2 - 4(\mathrm{AC}-\mathrm{B}^2)(\mathrm{BD}-\mathrm{C}^2) = 0.$$

Or the envelope is, in general or what is the same thing for a system of the class (A), a developable of the eighth order. For a system of the class (B) the equation contains as a factor, the square of the linear function which equated to zero is the equation of the plane of contact, or the envelope is in this case a Developable of the sixth order. And in the case of a system of the class (C) the equation contains as a factor the cube of this linear function, or the envelope is a developable of the fifth order only.

A. We may take for the two surfaces the reciprocals (with respect to $x^2+y^2+z^2+w^2=0$) of the equations made use of in determining the Intersect-Developable. The equations of these reciprocals are

$$bcdx^2 + cday^2 + dabz^2 + abcw^2 = 0,$$

$$b'c'd'x^2 + c'd'a'y^2 + d'a'b'z^2 + a'b'c'w^2 = 0;$$

and it is clear from the form of them (as compared with the equations of the surfaces of which they are the reciprocals) that $x=0$, $y=0$, $z=0$, $w=0$, are still the equations of the conjugate planes. We have, introducing the numerical factor 3 to avoid fractions,

$$\begin{aligned}3\{&(b+kb')(c+kc')(d+kd')x^2 + (c+kc')(d+kd')(a+ka')y^2 \\ &+ (d+kd')(a+ka')(b+kb')z^2 + (a+ka')(b+kb')(c+kc')w^2\} \\ &= \mathrm{A} + 3\mathrm{B}k + 3\mathrm{C}k^2 + \mathrm{D}k^3,\end{aligned}$$

which determine the values of A, B, C, D.

We have in fact

$$\mathrm{A} = 3\,(bcdx^2 + cday^2 + dabz^2 + abcw^2)$$
$$\mathrm{B} = (b'cd + bc'd + bcd')\,x^2 + \ldots\ldots$$
$$\mathrm{C} = (bc'd' + b'cd' + b'c'd)\,x^2 + \ldots\ldots$$
$$\mathrm{D} = 3\,(b'c'd'x^2 + c'd'a'y^2 + d'a'b'z^2 + a'b'c'w^2).$$

And these values give (with the same signification as before of f, y, h, l, m, n)

$$\begin{aligned}
2\,(\mathrm{AC} - \mathrm{B}^2) &= Aa^2 + Bb^2 + Cc^2 + 2Fbc + 2Gca + 2Hab \\
&\quad + 2Lad + 2Mbd + 2Ncd + Dd^2, \\
2\,(\mathrm{BD} - \mathrm{C}^2) &= Aa'^2 + Bb'^2 + Cc'^2 + 2Fb'c' + 2Gc'a' + 2Ha'b' \\
&\quad + 2La'd' + 2Mb'd' + 2Nc'd' + Dd'^2, \\
\mathrm{AD} - \mathrm{BC} &= Aaa' + Bbb' + Ccc' \\
&\quad + F(bc' + b'c) + G(ca' + c'a) + H(ab' + a'b) \\
&\quad + L(ad' + a'd) + M(bd' + b'd) + N(cd' + c'd) + Ddd',
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{A} &= n^2y^4 + m^2x^4 + f^2w^4 + 2fm\,z^2w^2 - 2fn\,y^2w^2 + 2nm\,y^2z^2, \\
\mathrm{B} &= l^2z^4 + n^2x^4 + g^2w^4 + 2gn\,x^2w^2 - 2gl\,z^2w^2 + 2ln\,z^2x^2, \\
\mathrm{C} &= m^2x^4 + l^2y^4 + h^2w^4 + 2hl\,y^2w^2 - 2hm\,x^2w^2 + 2ml\,x^2y^2, \\
\mathrm{F} &= l^2y^2z^2, \\
\mathrm{G} &= m^2z^2x^2, \\
\mathrm{H} &= n^2x^2y^2, \\
\mathrm{L} &= f^2x^2w^2, \\
\mathrm{M} &= g^2y^2w^2, \\
\mathrm{N} &= h^2z^2w^2, \\
\mathrm{D} &= f^2x^4 + g^2y^4 + h^2z^4 - 2gh\,y^2z^2 - 2hf\,z^2x^2 - 2fg\,y^2z^2.
\end{aligned}$$

And then

$$\begin{aligned}
&4\,(\mathrm{AC} - \mathrm{B}^2)\,(\mathrm{BD} - \mathrm{C}^2) - (\mathrm{AD} - \mathrm{BC})^2 \\
&\quad = (BC - F^2)\,f^2 + (CA - G^2)\,g^2 + (AB - H^2)\,h^2 \\
&\qquad\qquad + (AD - L^2)\,l^2 + (BD - M^2)\,m^2 + (CD - N^2)\,n^2 \\
&+ 2\,(GH - AF)\,gh + 2\,(HF - BG)\,hf + 2\,(FG - CH)\,fg \\
&- 2\,(MN - DF)\,mn - 2\,(NL - DG)\,nl - 2\,(LM - DH)\,lm \\
&+ 2\,(AM - LH)\,lh + 2\,(BN - MF)\,mf + 2\,(CL - NG)\,ng \\
&- 2\,(AN - LG)\,lg - 2\,(BL - MH)\,mh - 2\,(CM - NF)\,nf \\
&+ 2\,(NH - MG)\,lf + 2\,(LF - NH)\,mg + 2\,(MG - LF)\,nh.
\end{aligned}$$

And substituting the values of A, B, &c. in this expression, the result after all reductions is

$$
\begin{aligned}
& f^2m^2n^2x^8 + g^2n^2l^2y^8 + h^2l^2m^2z^8 + f^2g^2h^2w^8 \\
& + 2gl^2n\,(mg - nh)\,y^6z^2 + 2hm^2l\,(nh - lf)\,z^6x^2 + 2fn^2m\,(lf - mg)\,x^6y^2 \\
& - 2hl^2m\,(mg - nh)\,y^2z^6 - 2fm^2n\,(nh - lf)\,z^2x^6 - 2gn^2l\,(lf - mg)\,x^2y^6 \\
& + 2f^2mn(mg - nh)\,x^6w^2 + 2g^2nl(nh - lf)\,y^6w^2 + 2h^2lm(lf - mg)\,z^6w^2 \\
& - 2f^2gh\,(mg - nh)\,x^2w^6 - 2fg^2h\,(nh - lf)\,y^2w^6 - 2fgh^2(lf - mg)\,z^2w^6 \\
& + f^2(l^2f^2 - 6ghmn)\,w^4x^4 + g^2(m^2g^2 - 6hfnl)\,w^4y^4 + h^2(n^2h^2 - 6lmfg)\,w^4z^4 \\
& + l^2(l^2f^2 - 6ghmn)\,y^4z^4 + m^2(m^2g^2 - 6hfnl)\,z^4x^4 + n^2(n^2h^2 - 6lmfg)\,x^4y^4 \\
& + 2gh\,(ghmn - 3f^2l^2)\,w^4y^2z^2 + 2hf(hfnl - 3g^2m^2)\,w^4z^2x^2 \\
& \qquad\qquad + 2fg\,(fglm - 3h^2n^2)\,w^4x^2y^2 \\
& + 2hm\,(ghmn - 3f^2l^2)\,z^4x^2w^2 + 2fn\,(hfnl - 3g^2m^2)\,x^4y^2w^2 \\
& \qquad\qquad + 2gl\,(fglm - 3h^2n^2)\,y^4z^2w^2 \\
& - 2gn\,(ghmn - 3f^2l^2)\,y^4x^2w^2 - 2hl\,(hfnl - 3g^2m^2)\,z^4y^2w^2 \\
& \qquad\qquad - 2fm\,(fglm - 3h^2n^2)\,x^4z^2w^2 \\
& - 2mn\,(ghmn - 3f^2l^2)\,x^4y^2z^2 - 2nl\,(hfnl - 3g^2m^2)\,y^4z^2x^2 \\
& \qquad\qquad - 2lm\,(fglm - 3h^2n^2)\,z^4x^2y^2 \\
& - 2\,(mg - nh)\,(nh - lf)\,(lf - mg)\,x^2y^2z^2w^2 = 0,
\end{aligned}
$$

which is therefore the equation of the envelope for this case. The equation may also be presented under the form

$$
\begin{aligned}
& w^2\Theta + (mnx^2 + nly^2 + lmz^2)^2 \\
& \qquad (f^2x^4 + g^2y^4 + h^2x^4 - 2ghy^2z^2 - 2hfz^2x^2 - 2fgx^2y^2) = 0\,;
\end{aligned}
$$

and there are probably other forms proper to exhibit the different geometrical properties of the surface, but with which I am not yet acquainted.

B. Here taking for the two surfaces the reciprocals of the equations made use of in determining the Intersect-Developable, the equations of these reciprocals are

$$
\begin{aligned}
n^2bx^2 + n^2ay^2 - abcw^2 + 2nabzw &= 0, \\
n'^2b'x^2 + n'^2a'y^2 - a'b'c'w^2 + 2n'a'b'zw &= 0,
\end{aligned}
$$

which are similar to the equations of the surfaces of which they are reciprocal, only z and w are interchanged, so that here $x = 0$, $y = 0$ are the single conjugate planes, $z = 0$ is an indeterminate plane passing through the single conjugate points, and $w = 0$ is the equation of the double conjugate plane or plane of contact.

The values of A, B, C, D are

$$A = 3\,(n^2bx^2 + n^2ay^2 - abcw^2 + 2nabzw),$$

$$B = (2nn'b + n^2b')\,x^2 + \ldots.$$

$$C = (2nn'b' + n'^2b)\,x^2 + \ldots.$$

$$D = 3\,(n'^2b'x^2 + n'^2a'y^2 - a'b'c'w^2 + 2n'a'b'zw).$$

Hence, using f, g, h, p, q, r in the same sense as before, we have for $2\,(AC - B^2)$, $2\,(BD - C^2)$, $(AD - BC)$ expressions of the same form as in the last case (p, q, r being written for l, m, n), but in which

$$A = f^2w^4 + 4q^2z^2w^2 + 8qry^2w^2 - 4fqzw^3,$$

$$B = g^2w^4 + 4p^2z^2w^2 + 8prx^2w^2 + gpzw^3,$$

$$C = h^2w^4,$$

$$F = -\,p^2y^2w^2,$$

$$G = -\,q^2x^2w^2,$$

$$H = 0,$$

$$L = 2pqy^2zw,$$

$$M = 2pqx^2zw,$$

$$N = -\,2h^2zw^3,$$

$$D = 2q^2x^4 + 2p^2y^4 + 4h^2z^2w^2 + 4pqx^2y^2 - 8qhx^2zw + 8phy^2zw + 2ghy^2w^2 + 2fhx^2w^2.$$

The substitution of these values gives after all reductions the result

$$\begin{aligned}
&f^2g^2h^2w^6 + 4\,(pf - qg)fgh^2zw^5 \\
&+ 4\,(r^2h^2 - 6pqgf)\,h^2z^2w^4 + 2\,(q^2g^2 + 2prfh)\,fhx^2w^4 \\
&\qquad + 2\,(p^2f^2 + 2qrgh)\,ghy^2w^4 \\
&- 16\,(pf - qg)\,z^3w^3 - 4\,(q^2g^2 - 4p^2f^2 - 6pqfg)\,qhx^2zw^3 \\
&\qquad - 4\,(p^2f^2 - 4q^2g^2 - 6pqfg)\,phy^2zw^3 \\
&+ 16p^2q^2h^2z^4w^2 - 8\,(pf + 4qg)\,q^2phx^2z^2w^2 - 8\,(qg + 4pf)\,pq^2hy^2z^2w^2 \\
&+ (q^2g^2 + 8prfh)\,q^2x^4w^2 + (p^2f^2 + 8qrgh)\,p^2y^4w^2 \\
&\qquad + 2\,(10r^2h^2 - pqfg)\,pqx^2y^2w^2 \\
&- 16p^2q^3hx^2z^3w + 16p^3q^2hy^2z^3w \\
&+ 4\,(4pf + 5qg)\,pq^3x^4zw - 4\,(4qg + 5pf)\,p^3qy^4zw \\
&\qquad - 4\,(pf - qg)\,p^2q^2x^2y^2zw \\
&+ 4p^2q^4x^4z^2 + 4p^4q^2y^4z^2 + 8p^3q^3x^2y^2z^2 \\
&+ 4pq^4rx^6 + 4p^4qry^6 + 12p^2q^3rx^4y^2 + 12p^3q^2rx^2y^4 = 0\,;
\end{aligned}$$

which is therefore the equation of the envelope for this case. This equation may be presented under the form

$$w\Psi + 4pq(qx^2 + py^2)^2(qrx^2 + rpy^2 + pqz^2) = 0,$$

and there are probably other forms which I am not yet acquainted with.

C. The reciprocals of the two surfaces made use of in determining the Intersect-Developable, although in reality a system of the same nature with the surfaces of which they are reciprocals, are represented by equations of a somewhat different form. There is no real loss of generality in replacing the two surfaces by the reciprocals of the cones $x^2 = 2yz$, $y^2 = 2zw$; or we may take the two conics

$$(x^2 - 2yz = 0,\ w = 0) \text{ and } (y^2 - 2zw = 0,\ x = 0),$$

for the surfaces of which the envelope has to be found, these conics being, it is evident, the sections by the planes $w = 0$ and $x = 0$ respectively of the cones the Intersect-Developable of which was before determined. The process of determining the envelope is however essentially different. Supposing the plane $\xi x + \eta y + \zeta z + \omega w = 0$ to be the equation of a tangent plane to the two conics (*i.e.* of a plane passing through a tangent of each of the conics). The condition of touching the first conic gives $\xi^2 - 2\eta\zeta = 0$, and that of touching the second conic gives $\eta^2 - 2\zeta\omega = 0$. We have therefore to find the envelope (in the ordinary sense of the word) of the plane $\xi x + \eta y + \zeta z + \omega w = 0$, in which the coefficients ξ, η, ζ, ω are variable quantities subject to the conditions

$$\xi^2 - 2\eta\zeta = 0, \quad \eta^2 - 2\zeta\omega = 0.$$

The result which is obtained without difficulty by the method of indeterminate multipliers, is

$$8zx^4 - 32z^2x^2w + 32z^3w^2 - 27wy^4 + 72wxy^2z - 4x^3y^2 = 0,$$

which may also be written under the form

$$8z(x^2 - 2zw)^2 - y^2\{9w(3y^2 - 8zx) + 4x^3\} = 0.$$

ON THE THEOREMS IN SPACE ANALOGOUS TO THOSE OF PASCAL AND BRIANCHON IN A PLANE.—PART II.

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At the time the theorems given in my former memoir* occurred to me, I was not (I confess) aware that any theorems analogous to those of Pascal and Brianchon had been published. Subsequently, as mentioned in the note at p. 44, (vol. IV.) I discovered other analogues, but I soon found that I had been anticipated by M. Chasles. Finally, almost immediately after writing the 'note' just alluded to, I discovered other analogous theorems. I purpose (if the requisite space can be afforded me) presenting the whole of these researches as continuations of my former memoir—retaining the title as indicating the chief object I had in view in my investigations, but modifying the plan so far as to allow me to introduce much interesting collateral matter.

The analogues to be given in this part are the following theorems (I.) and (II.).

I. *If eight planes intersect, three and three in order, in eight points, such that* EVERY *surface of the second degree passing through seven of them shall also pass through the eighth point, then* EVERY *straight line resting upon three of the four straight lines in which the opposite planes intersect, will rest upon the fourth*; or in other words, *the opposite planes will intersect in four straight lines, belonging to the same system of generators, in an hyperboloid of one sheet.*

II. *If eight planes be such that* EVERY *surface of the second degree touching seven of them shall also touch the eighth, then* EVERY *straight line resting upon three of the four straight lines joining the opposite points in which these planes intersect three and three in order, will also rest upon the fourth*; or in other words, *the four straight lines joining the opposite points will lie in an hyperboloid of one sheet, and will belong to the same system of generators.*†

I proceed to establish these theorems, considering them in order.

Let $t = 0$, $u = 0$, $v = 0$, $w = 0$, be the equations to four successive planes; then, supposing t, u, v and w to have been multiplied by the proper constants, we may denote the

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* *Journal*, (new series) vol. IV. p. 26.

† (I) and (II) may be regarded as properties of the twisted (*gauche*) octagon.

equations to the planes that are opposite to these by

$$\left.\begin{aligned} t' &= t + \lambda u + \mu v + \nu w = 0 \\ u' &= u + \lambda t + \rho v + \sigma' w = 0 \\ v' &= v + \mu t + \rho' u + \tau w = 0 \\ \text{and} \quad w' &= w + \nu t + \sigma u + \tau' v = 0 \end{aligned}\right\} \ldots\ldots\ldots (1).$$

It is easy to see that the equation to any surface of the second degree passing through the four points $(wt'u')$, $(t'u'v')$, $(u'v'w')$ and $(v'w't)$* may be denoted by

$$Atu' + Bv'w + Ct'w' + Du'v' + Et'v' + Fu'w' = 0 \,..(2).$$

Now if this surface pass through the point (vwt'), we must have, (2),

$$At + Dv' + Fw' = 0,$$

also, (1), $t = -\lambda u$, $v' = \mu t + \rho' u = (\rho' - \lambda\mu)\, u$, and $w' = \nu t + \sigma u = (\sigma - \lambda\nu)\, u$;

therefore $\quad -\lambda A + (\rho' - \lambda\mu)\, D + (\sigma - \lambda\nu)\, F = 0 \,\ldots.(3).$

Similarly, if the surface pass through the point $(w'tu)$, we must have

$$-\tau' B + (\rho - \sigma'\tau')\, D + (\mu - \nu\tau')\, E = 0 \ldots\ldots(4).$$

Moreover we immediately see from (1, 2) that if the surface pass through the point (uvw), then will

$$\lambda A + \nu C + \lambda\mu D + \mu E + \lambda\nu F = 0 \ldots\ldots\ldots(5),$$

and if it pass through the point (tuv) we shall have

$$\tau B + \nu C + \sigma'\tau D + \nu\tau E + \sigma' F = 0 \ldots\ldots\ldots(6).$$

Eliminate A from (3, 5) and B from (4, 6), therefore

$$\nu C + \rho' D + \mu E + \sigma F = 0 \ldots\ldots\ldots\ldots(7),$$

and $$\nu C + \frac{\rho\tau}{\tau'} D + \frac{\mu\tau}{\tau'} E + \sigma' F = 0 \ldots\ldots\ldots\ldots(8).$$

The equations (3, 4, 7, 8) are the conditions that must exist in order that the surface (2) may pass through all the eight points, and it hence easily follows that if every surface of the second degree passing through seven of the points shall also

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* To avoid circumlocution, I use the following abbreviations. The plane t means the plane whose equation is $t = 0$; the straight line (tu) the straight line in which the planes t and u intersect; the point (tuv) the point in which the planes t, u, and v intersect; and the tetrahedron $(tuvw)$ the tetrahedron whose faces coincide with the planes t, u, v, and w.

pass through the eighth, the equations (7) and (8) must be identical: this requires $\rho = \rho'$, $\sigma = \sigma'$, and $\tau = \tau'$. Hence the equations (1) become

$$\left.\begin{aligned} t' &= t + \lambda u + \mu v + \nu w = 0 \\ u' &= u + \lambda t + \rho v + \sigma w = 0 \\ v' &= v + \mu t + \rho u + \tau w = 0 \\ w' &= w + \nu t + \sigma u + \tau v = 0 \end{aligned}\right\} \quad \ldots\ldots\ldots (9).$$

Consequently the lines (tt'), (uu'), (vv') and (ww') all lie in the hyperboloid of one sheet whose equation is

$$\begin{aligned} &\lambda\mu\nu t^2 + \lambda\rho\sigma u^2 + \mu\rho\tau v^2 + \nu\sigma\tau w^2 + (\mu\sigma + \nu\rho)(\lambda tu + \tau vw) \\ &\quad + (\lambda\tau + \nu\rho)(\mu tv + \sigma uw) + (\lambda\tau + \mu\sigma)(\nu tw + \rho uv) = 0 \,.\,.(10), \end{aligned}$$

for this equation may be written

$$\begin{aligned} &\{\lambda\mu\nu t + (\mu\sigma + \nu\rho)\lambda u + (\lambda\tau + \nu\rho)\mu v + (\lambda\tau + \mu\sigma)\nu w\}\, t \\ &\qquad\qquad + (\lambda u + \mu v + \nu w)(\rho\sigma u + \rho\tau v + \sigma\tau w) = 0, \end{aligned}$$

and, (9), this is evidently satisfied by $t = 0$ and $t' = 0$. The theorem (I) is therefore true.

Moreover if eight points be such that every surface of the second degree passing through seven of them shall also pass through the eighth, their reciprocals will evidently be a system of eight planes such that every surface of the second degree touching seven of them shall also touch the eighth, and conversely: also if a number of straight lines lie in an hyperboloid of one sheet, their reciprocals will also lie in an hyperboloid of one sheet. Hence (II) is merely the reciprocal of (I.).

Conversely,

III. *Let eight planes intersect, three and three in order, in eight points. If the opposite planes intersect in four straight lines, belonging to the same system of generators, in an hyperboloid of one sheet,* EVERY *surface of the second degree passing through seven of the points will pass through the eighth.*

IV. *If the four straight lines joining the opposite points in which eight planes intersect three and three in order, lie in an hyperboloid of one sheet, and belong to the same system of generators, then* EVERY *surface of the second degree touching seven of the planes will also touch the eighth.*

It will be sufficient to establish the former of these two theorems; for the latter is obviously the reciprocal theorem.

It is evident that each of the planes mentioned in (III) is a tangent plane to the hyperboloid; hence, if $t = 0$, $u = 0$,

$v = 0$, and $w = 0$ be the equations to four successive planes, the hyperboloid will touch these four planes. Now I have shewn in the *Mathematician** that, supposing t, u, v and w to have been multiplied by arbitrary constants, the general equation to surfaces of the second degree touching the four planes just mentioned is

$$t^2 + u^2 + v^2 + w^2 + \left(\frac{m}{n} + \frac{n}{m}\right)(tu + vw) + \left(\frac{p}{m} + \frac{m}{p}\right)(tv + uw)$$
$$+ \left(\frac{n}{p} + \frac{p}{n}\right)(tw + uv) = 0 \ldots . (11);$$

and hence if the constants in this equation be properly assumed, it will denote the hyperboloid.

Let $t' = 0$, $u' = 0$, $v' = 0$, and $w' = 0$ be the equations to the other four planes, t' being opposite to t, &c. To find the equations to the straight line in which the planes t and t' intersect, we must (since they intersect in the hyperboloid) put $t = 0$ in (11), which is thus reduced to

$$(pu + nv + mw)\left(\frac{u}{p} + \frac{v}{n} + \frac{w}{m}\right) = 0;$$

hence $t = 0$, and either $pu + nv + mw = 0$, or $\frac{u}{p} + \frac{v}{n} + \frac{w}{m} = 0$, denote the straight line (tt'): but it is evidently immaterial which of the latter equations is taken; hence the straight line may be denoted by $t = 0$, and $pu + nv + mw = 0$, and consequently t' must $= at + pu + nv + mw$, where a is a constant. In a similar manner it may be shewn that u', the plane opposite to u, must be denoted by one of the two,

$$bu + pt + mv + nw = 0, \text{ or } bu + \frac{t}{p} + \frac{v}{m} + \frac{w}{n} = 0;$$

but the latter must be rejected, for the equations

$$at + pu + nv + mw = 0, \text{ and } bu + \frac{t}{p} + \frac{v}{m} + \frac{w}{n} = 0,$$

being satisfied by $t = u = nv + mw = 0$, it would follow that the straight lines (tt') and (uu') intersect, and could not therefore belong to the same system of generators of the hyperboloid; consequently we must have $u' = bu + pt + mv + nw$. Similarly we find $v' = cv + nt + mu + pw$ and $w' = ew + mt + nu + pv$.

* The reader will find it necessary to refer both to vol. II. p. 261, and vol. III. p. 279.

Hence the equations to the remaining four planes may be denoted as follows:

$$\left.\begin{aligned} t' &= at + pu + nv + mw = 0 \\ u' &= bu + pt + mv + nw = 0 \\ v' &= cv + nt + mu + pw = 0 \\ \text{and} \quad w' &= ew + mt + nu + pv = 0 \end{aligned}\right\} \ldots\ldots\ldots(12).$$

In these equations, write $\frac{t}{\sqrt{a}}$, $\frac{u}{\sqrt{b}}$, $\frac{v}{\sqrt{c}}$, $\frac{w}{\sqrt{e}}$, $\sqrt{a}\,t'$, $\sqrt{b}\,u'$, $\sqrt{c}\,v'$, $\sqrt{e}\,w'$, λ, μ, ν, ρ, σ, τ for t, u, v, w, t', u', v', w' $\frac{p}{\sqrt{ab}}$, $\frac{n}{\sqrt{ac}}$, $\frac{m}{\sqrt{ae}}$, $\frac{m}{\sqrt{bc}}$, $\frac{n}{\sqrt{be}}$, $\frac{p}{\sqrt{ce}}$, respectively; and then (12) will coincide with (9). If therefore we assume $t = 0$, $u = 0$, $v = 0$, $w = 0$, and the equations (9) as the equations to the eight planes mentioned in (III), and repeat the investigation by which (I) was established (only using (9) instead of (1),) we shall find that the two conditions corresponding to (7) and (8) are identical; and hence every surface of the second degree passing through seven of the points in which the eight planes intersect three and three in order, will pass through the eighth point.

It will be seen from the preceding investigation that if four lines (tt'), (uu'), (vv') and (ww') all lie in the same hyperboloid of one sheet, and belong to the same system of generators, then t', u', v', and w', as expressed in terms of t, u, v, and w, may be denoted either by (9) or (12).

The following properties are intimately connected with the preceding theorems.

V.* *If two tetrahedra be such that the angular points of each are, with respect to a certain surface of the second degree, the poles of the opposite faces, then* EVERY *surface of the second degree touching seven of the faces will touch the eighth face, and* EVERY *surface of the second degree passing through seven of the angular points will also pass through the eighth angular point.*

Let $t = 0$, $u = 0$, $v = 0$, $w = 0$ be the equations to the faces of one of the tetrahedra, and

$$\left.\begin{aligned} t' &= at + bu + cv + ew = 0 \\ u' &= a_1t + b_1u + c_1v + e_1w = 0 \\ v' &= a_2t + b_2u + c_2v + e_2w = 0 \\ w' &= a_3t + b_3u + c_3v + e_3w = 0 \end{aligned}\right\} \ldots\ldots\ldots(13)$$

be the equations to the faces of the other tetrahedron.

[* Given by Hesse in the twentieth volume of *Crelle*, in a memoir 'De curvis et superficiebus secundi gradus.—A. C.]

Since the angular points of the tetrahedron $(t'u'v'w')$ are, with respect to the given surface (Σ) of the second degree, the poles of the opposite faces, the equation of Σ must be of the form

$$ht'^2 + ku'^2 + lv'^2 + mw'^2 = 0 \ldots\ldots\ldots (14).$$

Similarly, the equation of Σ must be of the form

$$h't^2 + k'u^2 + l'v^2 + m'w^2 = 0 \ldots\ldots\ldots (15).$$

Substitute the values of t', u', v', w', as given in (13), in (14), and make the resulting equation identical in form with (15); that is, equate the coefficients of tu, tv, tw, uv, uw, and vw to zero; therefore

$$\left.\begin{array}{l} hab + ka_1b_1 + la_2b_2 + ma_3b_3 = 0 \\ hac + ka_1c_1 + la_2c_2 + ma_3c_3 = 0 \\ hae + ka_1e_1 + la_2e_2 + ma_3e_3 = 0 \\ hbc + kb_1c_1 + lb_2c_2 + mb_3c_3 = 0 \\ hbe + kb_1e_1 + lb_2e_2 + mb_3e_3 = 0 \\ hce + kc_1e_1 + lc_2e_2 + mc_3e_3 = 0 \end{array}\right\} \ldots\ldots (16).$$

These equations enable us to find the three ratios $\frac{k}{h}$, $\frac{l}{h}$, $\frac{m}{h}$, and to deduce the three relations which must exist among the constants in (13) in order that the two tetrahedra may be such that the angular points of each may be the poles, relative to a certain surface of the second degree, of the opposite faces. The equation to this surface will of course be got by substituting the values of $\frac{k}{h}$, $\frac{l}{h}$, and $\frac{m}{h}$ in (14).

We have next to obtain the conditions required in order that every surface of the second degree touching seven of the faces may also touch the eighth face. So far as I can see, it would be no easy matter to obtain these conditions directly, but we may evade this difficulty in the following manner.

The conditions sought are evidently the very same as those required in order that every surface of the second degree passing through seven of the poles (relative to any surface of the second degree) of the faces of the tetrahedra may also pass through the eighth pole. Let us therefore take the poles of the faces relative to the surface whose equation is

$$t^2 + u^2 + v^2 = w^2;$$

the angular points of the tetrahedron $(tuvw)$ will be the poles

of its own faces, while the poles of the faces (13) will be denoted as under:

$$\left.\begin{array}{l} \dfrac{t}{a} = \dfrac{u}{b} = \dfrac{v}{c} = -\dfrac{w}{e} \\[2ex] \dfrac{t}{a_1} = \dfrac{u}{b_1} = \dfrac{v}{c_1} = -\dfrac{w}{e_1} \\[2ex] \dfrac{t}{a} = \dfrac{u}{b_2} = \dfrac{v}{c_2} = -\dfrac{w}{e_2} \\[2ex] \text{and} \quad \dfrac{t}{a_3} = \dfrac{u}{b_3} = \dfrac{v}{c_3} = -\dfrac{w}{e_3} \end{array}\right\} \ldots\ldots\ldots\ldots (17);$$

and we have now to obtain the conditions required that every surface of the second degree passing through seven of these poles may also pass through the eighth.

The equation to every surface of the second degree passing through the first four poles (that is, through the angular points of the tetrahedron $(tuvw)$,) is

$$Atu + Btv + Ctw + Duv + Euw + Fvw = 0 \ldots . (18);$$

and in order that this surface may pass through each of the points (17), we must have

$$Aab + Bac - Cae + Dbc - Ebe - Fce = 0,$$
$$Aa_1b_1 + Ba_1c_1 - Ca_1e_1 + Db_1c_1 - Eb_1e_1 - Fc_1e_1 = 0,$$
$$Aa_2b_2 + Ba_2c_2 - Ca_2e_2 + Db_2c_2 - Eb_2e_2 - Fc_2e_2 = 0,$$
and $$Aa_3b_3 + Ba_3c_3 - Ca_3e_3 + Db_3c_3 - Eb_3e_3 - Fc_3e_3 = 0.$$

Now any three of these equations must imply the fourth; hence, multiplying them by H, K, L, and M, respectively, and equating the coefficients of A, B, &c. to zero, we have

$$\left.\begin{array}{l} Hab + Ka_1b_1 + La_2b_2 + Ma_3b_3 = 0 \\ Hac + Ka_1c_1 + La_2c_2 + Ma_3c_3 = 0 \\ Hae + Ka_1e_1 + La_2e_2 + Ma_3e_3 = 0 \\ Hbc + Kb_1c_1 + Lb_2c_2 + Mb_3c_3 = 0 \\ Hbe + Kb_1e_1 + Lb_2e_2 + Mb_3e_3 = 0 \\ Hce + Kc_1e_1 + Lc_2e_2 + Mc_3e_3 = 0 \end{array}\right\} \ldots\ldots (19),$$

and the elimination of H, K, L, and M from these equations will determine the conditions required. But since the equations (19) only differ from (16) in having H, K, L, and M instead of h, k, l, and m, it is evident that these conditions are satisfied, and consequently every surface of the second

degree touching seven of the faces will also touch the eighth face.

Again, through seven of the angular points of the tetrahedra describe any surface S of the second degree; take, with respect to the surface Σ, the reciprocals of the angular points and of S, and there result the faces and a surface S' touching seven of these faces: hence, as we have just seen, S' will touch the eighth face, and consequently the reciprocal surface S will pass through the eighth angular point; that is, every surface of the second degree passing through seven of the angular points will pass through the eighth.

From the preceding investigation we also infer:

VI. *If two tetrahedra be such that* EVERY *surface of the second degree touching seven of the faces will also touch the eighth face, then* EVERY *surface of the second degree passing through seven of the angular points will pass through the eighth angular point: also there is a certain surface (real or unreal) of the second degree with respect to which the angular points of each tetrahedron are the poles of the opposite faces.*

Moreover the reciprocal of the last theorem is evidently as follows:

VII. *If two tetrahedra be such that* EVERY *surface of the second degree passing through seven of the angular points will also pass through the eighth; then* EVERY *surface of the second degree touching seven of the faces will touch the eighth face. Also there is a certain surface, &c.* (as in VI.).

The following properties (VIII.) and (IX.) may be appropriately added here.

VIII. EVERY *surface of the second degree passing through seven of the angular points of a hexahedron will also pass through the eighth angular point.*

Let AB, AC and AD be three contiguous edges of the hexahedron, and $t = 0$, $u = 0$, $v = 0$, the equations to the three faces CAD, BAD, and BAC, respectively: also let $s = 0$ be the equation to the plane which passes through the points B, C, and D; hence the equation to any surface of the second degree passing through the points A, B, C, and D (which are the angular points of the tetrahedron $(stuv)$,) will be

$$gst + hsu + ksv + luv + mtu + ntv = 0 \dots\dots(20).$$

Again, since the faces of the hexahedron that are opposite to t, u, and v, pass through the points B, C, and D respec-

tively, their equations must be of the forms

$$t' = s + au + bv = 0,$$

$$u' = s + ct + ev = 0,$$

and $$v' = s + ft + iu = 0.$$

Hence the angular point $(tu'v')$ opposite to B, will be denoted by $t = 0$, $u = -\frac{s}{i}$, and $v = -\frac{s}{e}$; and these values substituted in (20) give $l = he + ki$. In like manner, by considering the angular points opposite to C and D, we get $n = gb + kf$, and $m = ga + hc$; and these values being substituted in (20), we find that the general equation to surfaces of the second degree passing through seven of the angular points of the hexahedron, is

$$gt(s + au + bv) + hu(s + ct + ev) + kv(s + ft + iu) = 0,$$

that is $$gtt' + huu' + kvv' = 0 \ldots\ldots\ldots\ldots (21).^*$$

Now this equation is likewise satisfied by $t' = u' = v' = 0$, and hence the surface also passes through the eighth angular point.

IX. Every *surface of the second degree touching seven of the faces of an octahedron will touch the eighth face.*

The poles, relative to any surface of the second degree, of the faces of an octahedron are evidently the angular points of a hexahedron. Let any surface (S) of the second degree be described to touch seven of the faces of the octahedron, and take the reciprocal of this system; we shall thus obtain a surface (S') of the second degree passing through seven of the angular points of a hexahedron; hence, (VIII.), S' passes through the eighth angular point, and consequently the eighth face of the octahedron, which is the polar of this angular point, will touch S, which is the reciprocal of S'.

From (VIII.) and (IX.) several theorems might, by aid of some of the previous propositions, be inferred, but I shall insert one only.

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* This is of course the general equation to surfaces of the second degree circumscribed about a hexahedron. In like manner, if $t = 0$, $u = 0$, $v = 0$, $w = 0$, $t' = 0$, $u' = 0$, $v' = 0$, and $w' = 0$, be the equations to the faces of an octahedron, t' being opposite to t, &c., then will

$$gtt' + huu' + kvv' + lww' = 0$$

be the general equation to surfaces of the second degree circumscribed about the octahedron; for at each angular point either t or $t' = 0$, either u or $u' = 0$, either v or $v' = 0$, and either w or $w' = 0$.

The faces of two tetrahedra whose corresponding edges intersect, will evidently form the faces of an octahedron; hence the following theorem immediately follows from (VI.) and (IX.).

X. *If the corresponding edges of two tetrahedra intersect,* EVERY *surface of the second degree touching seven of the faces will touch the eighth face, and* EVERY *surface of the second degree passing through seven of the angular points will pass through the eighth angular point. Also there is a certain surface (real or unreal) of the second degree with respect to which the angular points of each tetrahedron are the poles of the opposite faces.*

I shall add a few observations, partly with the view of calling attention to what remains to be done on a subject partially treated in this Paper.

In Plane Geometry we may ask, What is the relation among six points on a conic section? and Pascal's theorem furnishes an answer to this question. In Solid Geometry we may ask *not fewer* than three questions (how many more I will not take upon me to say), each having more or less analogy to the preceding:

1. What is the relation among ten points in a surface of the second degree?

2. What are the relations among nine points in space such that *every* surface of the second degree passing through eight of them may also pass through the ninth? or, in other words, what are the relations among nine points situated on the curve in which two surfaces of the second degree intersect?

3.* What are the relations among eight points in space such that *every* surface of the second degree passing through seven of them may also pass through the eighth?†

[* This has been answered by Hesse in the memoir quoted at the foot of p. 62; and more simply in a memoir in the 26th volume, "Ueber die lineale Construction des achten Schnittpunktes, &c." From a note in this last memoir, Hesse seems to be in possession of a construction for the curve of intersection of two surfaces of the second order when eight points are given—the construction however not being linear but requiring a circle. I am not aware that this construction has been published. It would depend on the form of it whether it was or was not precisely an answer to question (2). (Hesse seems also to be in possession of a construction for the ninth point of intersection of two plane curves of the third order when eight points are given, but this has not I believe been published.) (1) has never been solved, although a surface of the second order may be constructed by means of nine given points. See last volume of *Math. Journal*.—A. C.]

† Of course three questions analogous to the above may be asked respecting surfaces of the second degree touching a system of ten, nine, or eight planes; and theorem (II) furnishes an answer to the last question.

It may be as well to observe, too, that we may ask, What is the relation

The first and second of these questions have been asked before, though, I believe, neither of them has yet been answered; but I am not aware that the third has even been so much as thought of. Although I have not been able to answer either of the first two questions, yet I have been more successful with the third, as theorem (I.) will shew. The three relations furnished by (I.) in order that eight points in space may be such that every surface of the second degree passing through seven of them may also pass through the eighth, may be stated as follows. Take the eight points in any order and join them by straight lines, thus forming a twisted octagon: through every two successive sides pass a plane, find the four straight lines in which the opposite planes intersect, and draw (any) three straight lines resting upon three of these four intersections: then if each of these three lines rests upon the fourth intersection, the eight points are such a system as supposed; otherwise they are not so.

In connection with the second question, it is to be remarked that though the curve in which two surfaces of the second degree intersect is in general of the fourth degree, yet in certain cases it may reduce to the third degree (or rather the section may consist of a straight line and a curve of the third degree), and then *six* points will be sufficient to determine the curve. In this case we may seek the relation among *seven* points on the curve, and this has, in fact, been already given by M. Chasles, in his '*Aperçu Historique... des Méthodes en Géométrie,*' Note XXXIII. This great geometer enunciates his theorems as follows.

When the vertices, a, b, c,...g, *of a twisted heptagon* (*eptagone gauche*) *are situated on a curve of double curvature of the third degree, the plane of any one of its angles* a, *and the planes of the two adjacent angles,* b *and* g, *will intersect the opposite sides* de, ef, *and* cd, *in three points, which are in a plane passing through the vertex of the first angle* a.

Conversely,

When a twisted heptagon is such that the plane of one of its angles and the planes of the two adjacent angles intersect the opposite sides in three points, which are in a plane passing through the vertex of the first angle; and that the same is true for one of the six other angles;

among ten straight lines in space which touch a surface of the second degree? but I have not sufficiently considered the subject to be able to say whether a system of straight lines (not all in one plane) can exist, such that every surface of the second degree touching all of them but one shall also touch that one.

then shall this likewise be true for each of the five remaining angles, and the seven vertices of the heptagon will be situated on a curve of double curvature of the third degree.

It is evident (as M. Chasles remarks) that this theorem enables us to construct the curve when six points in it are given; and in this respect it closely resembles Pascal's theorem.*

March 26, 1849.

ON THE BIFOCAL CHORDS OF SURFACES OF THE SECOND ORDER.

By JOHN Y. RUTLEDGE, A.B., Trinity College, Dublin.

1. IN a very able paper, published in the *Transactions of the Royal Irish Academy*, MacCullagh has established the modular method of generating surfaces of the second order. He has shewn that these surfaces may be regarded as the "loci" of a point whose distance from a fixed point (a focus) bears a constant ratio to its distance from a fixed right line (a directrix) measured parallel to a fixed plane (a directive or cyclic plane). This beautiful memoir has invested with a new interest the Geometry of Surfaces of the Second Order, and has led to the discovery of many properties before unsuspected. The focal and dirigent curves in their real connexion with each surface were thus for the first time brought into evidence: upon this part of our subject, however, it is needless to dwell longer, as MacCullagh's memoir is now generally well known. The connexion between every

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* I cannot forbear expressing a doubt, though with some diffidence, as to the correctness of one of the theorems which M. Chasles has given in Note XXXIII. It is as follows:

The locus of the vertex of the cone of the second degree which passes through six given points in space, is the curve of double curvature of the third degree determined by (*i.e.* passing through) these six points.

Now the analysis to which I have subjected the problem leads me to conclude that the locus is a *surface* of the fourth degree, which, as M. Chasles's own investigation shews, must pass through the curve of double curvature of the third degree determined by the six points. In fact, Chasles only proves the following theorem:—Every cone which has its vertex on a curve of double curvature of the third degree, and which passes through the curve, is of the second degree. And his error seems to have arisen from momentarily imagining that every cone of the second degree passing through the six points, will also pass through the curve of the third degree determined by these points.

point upon the surface and a point upon the focal curve having been thus indicated, the question naturally arose, as to the consequences that would follow from the connexion, by a right line, of any point upon the surface with two points, one on *each* focal curve. The values for the lengths of the bifocal chords (*i.e.* right lines which pass through a point on each focal curve, and are terminated on either side by the surface) MacCullagh determined; but farther (so far as I am aware) he did not prosecute the subject. It is this portion of the modular theory which the present article seeks to complete, and we shall see that it brings into evidence (in addition to many theorems which we believe to be new) the most important known theorems of the geometry of surfaces of the second order. Between them it also indicates a natural order of dependence. The want of which has hitherto considerably detracted from their utility and beauty, a very difficult theorem (not unfrequently) having been selected for the proof of a theorem naturally anterior and naturally much more simple.

2. Let the equation of the ellipsoid be

$$\frac{x^2}{\rho^2} + \frac{y^2}{\rho^2 - b^2} + \frac{z^2}{\rho^2 - c^2} = 1.*$$

[* The following proof is somewhat simpler.

Let X, Y, Z be the coordinates of any point in the chord, then the equation of the chord may be written

$$x = X + s\cos\phi, \quad y = Y + s\cos\psi, \quad z = Z + s\cos\omega,$$

where s is indeterminate. Write for shortness

$$\frac{1}{R^2} = \frac{\cos^2\phi}{\rho^2} + \frac{\cos^2\psi}{\rho^2 - b^2} + \frac{\cos^2\omega}{\rho^2 - c^2},$$

$$\Omega = \frac{X\cos\phi}{\rho^2} + \frac{Y\cos\psi}{\rho^2 - b^2} + \frac{Z\cos\omega}{\rho^2 - c^2},$$

$$Y\cos\omega - Z\cos\psi = \xi, \quad Z\cos\phi - X\cos\omega = \eta, \quad X\cos\psi - Y\cos\psi = \zeta.$$

(R is evidently the semidiameter parallel to the chord.)

Substituting in the equation to the surface, we have, after some reductions, the following quadratic equation to determine s:

$$\left(\frac{s}{R^2} + \Omega\right)^2 = \frac{1}{\rho^2.(\rho^2 - b^2)(\rho^2 - c^2)} \{(\rho^2 - b^2)(\rho^2 - c^2)\cos^2\phi + \rho^2.(\rho^2 - b^2)\cos^2\psi$$
$$+ \rho^2.(\rho^2 - c^2)\cos^2\omega - \rho^2\xi^2 - (\rho^2 - b^2)\eta^2 - (\rho^2 - c^2)\zeta^2\}.$$

It is easily seen that, whatever be the value of ρ,

$$(\rho^2 - b^2)(\rho^2 - c^2)\cos^2\phi + \rho^2.(\rho^2 - b^2)\cos^2\psi + \rho^2(\rho^2 - c^2)\cos^2\omega$$
$$- \rho^2\xi^2 - (\rho^2 - b^2)\eta^2 - (\rho^2 - c^2)\zeta^2 = (\rho^2 - a^2)(\rho^2 - a'^2).$$

From any point upon the surface of this ellipsoid let a right line, the equations of which are

$$y = mx + n, \quad z = m'x + n',$$

be drawn common tangent to the two confocal surfaces

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - b^2} + \frac{z^2}{a^2 - c^2} = 1, \quad \frac{x^2}{a'^2} + \frac{y^2}{a'^2 - b^2} + \frac{z^2}{a'^2 - c^2} = 1,$$

where, if ϕ, ψ, ω be the angles which the right line makes with the axes of (x, y, z), we shall obviously have

$$m = \frac{\cos\psi}{\cos\phi} \text{ and } m' = \frac{\cos\omega}{\cos\phi}.$$

These values for y and z let us substitute in the equation of the ellipsoid. Then

$$\frac{x^2}{\rho^2} + \frac{m^2x^2 + 2mnx + n^2}{\rho^2 - b^2} + \frac{m'^2x^2 + 2m'n'x + n'^2}{\rho^2 - c^2} = 1,$$

$$x^2 + 2Ax = B^2, \quad x = A \pm \sqrt{(A^2 + B^2)},$$

$$\text{where } A = \frac{\dfrac{mn}{\rho^2 - b^2} + \dfrac{m'n'}{\rho^2 - c^2}}{\dfrac{1}{\rho^2} + \dfrac{m^2}{\rho^2 - b^2} + \dfrac{m'^2}{\rho^2 - c^2}}, \quad B^2 = \frac{1 - \dfrac{n^2}{\rho^2 - b^2} - \dfrac{n'^2}{\rho^2 - c^2}}{\dfrac{1}{\rho^2} + \dfrac{m^2}{\rho^2 - b^2} + \dfrac{m'^2}{\rho^2 - c^2}}$$

$$A^2 + B^2 = \frac{\dfrac{1}{\rho^2} + \dfrac{m^2}{\rho^2 - b^2} + \dfrac{m'^2}{\rho^2 - c^2} - \dfrac{n^2}{\rho^2(\rho^2 - b^2)} - \dfrac{n'^2}{\rho^2(\rho^2 - c^2)} - \dfrac{(mn' - m'n)^2}{(\rho^2 - b^2)(\rho^2 - c^2)}}{\left\{\dfrac{1}{\rho^2} + \dfrac{m^2}{\rho^2 - b^2} + \dfrac{m'^2}{\rho^2 - c^2}\right\}^2}.$$

(This furnishes therefore two equations which, with

$$\xi \cos\phi + \eta \cos\psi + \zeta \cos\omega = 0,$$

determine ξ, η, ζ). Hence, putting for shortness,

$$\Delta^2 = \frac{(\rho^2 - a^2)\,(\rho^2 - a'^2)}{\rho^2(\rho^2 - b^2)\,(\rho^2 - c^2)},$$

we have $$s = R^2(-\Omega \pm \Delta).$$

Substituting in the values of x, y, z, a simple reduction gives

$$x = R^2\left\{\frac{\cos\psi}{\rho^2 - b^2}\zeta - \frac{\cos\omega}{\rho^2 - c^2}\eta \pm \Delta\cos\phi\right\},$$

$$y = R^2\left\{\frac{\cos\omega}{\rho^2 - c^2}\xi - \frac{\cos\phi}{\rho^2}\zeta \pm \Delta\cos\psi\right\},$$

$$z = R^2\left\{\frac{\cos\phi}{\rho^2}\eta - \frac{\cos\psi}{\rho^2 - b^2}\xi \pm \Delta\cos\omega\right\},$$

which are the formula at the foot of p. 73.—A. C.]

Hence $A^2 + B^2 = \dfrac{(\rho^2 - b^2)(\rho^2 - c^2) + \rho^2(\rho^2 - c^2)\,m^2 + \rho^2(\rho^2 - b^2)\,m'^2 - (\rho^2 - c^2)\,n^2 - (\rho^2 - b^2)\,n'^2 - \rho^2(mn' - m'n)^2}{\rho^2(\rho^2 - b^2)(\rho^2 - c^2)\left(\dfrac{1}{\rho^2} + \dfrac{m^2}{\rho^2 - b^2} + \dfrac{m'^2}{\rho^2 - c^2}\right)^2}$.

But since the chord is a common tangent to two confocal surfaces, it is manifest that for each of them we have

$$A^2 + B^2 = 0.$$

Hence
$$(a^2 - b^2)(a^2 - c^2) + a^2(a^2 - c^2)\,m^2 + a^2(a^2 - b^2)\,m'^2 - (a^2 - c^2)\,n^2 - (a^2 - b^2)\,n'^2 - \{a^2(mn' - m'n)^2\} = 0,$$
$$(a'^2 - b^2)(a'^2 - c^2) + a'^2(a'^2 - c^2)\,m^2 + a'^2(a'^2 - b^2)\,m'^2 - (a'^2 - c^2)\,n^2 - (a'^2 - b^2)\,n'^2 - \{a'^2(mn' - m'n)^2\} = 0.$$

Subtract the last two equations one from the other, and we shall have

$$n^2 + n'^2 + (mn' - m'n)^2 = \frac{(a^2 - b^2)(a^2 - c^2) - (a'^2 - b^2)(a'^2 - c^2) + \{a^2(a^2 - c^2) - a'^2(a'^2 - c^2)\}\,m^2 + \{a^2(a^2 - b^2) - a'^2(a'^2 - b^2)\}\,m'^2}{(a^2 - a'^2)}.$$

Multiply the first of the same equations by a'^2 and the second by a^2, and again subtract, and we shall have

$$c^2n^2 + b^2n'^2 = \frac{a'^2(a^2 - b^2)(a^2 - c^2) - a^2(a'^2 - b^2)(a'^2 - c^2) + \{a'^2a^2(a^2 - c^2) - a^2a'^2(a'^2 - c^2)\}m^2 + \{a'^2(a^2 - b^2)(a^2 - c^2) - a^2(a'^2 - b^2)(a'^2 - c^2)\}m'^2}{(a^2 - a'^2)}.$$

Now since we obviously have

$$A^2 + B^2 = \frac{(\rho^2 - b^2)(\rho^2 - c^2) + \rho^2(\rho^2 - c^2)\,m^2 + \rho^2(\rho^2 - b^2)\,m'^2 - \{n^2 + n'^2 + (mn' - m'n)^2\}\,\rho^2 + c^2n^2 + b^2n'^2}{\rho^2(\rho^2 - b^2)(\rho^2 - c^2)\left(\dfrac{1}{\rho^2} + \dfrac{m^2}{\rho^2 - b^2} + \dfrac{m'^2}{\rho^2 - c^2}\right)};$$

substitute their values for $n^2 + n'^2 + (mn' - m'n)^2$ and $b^2n'^2 + c^2n^2$; and remembering that

$$a^2(a^2 - c^2) = \rho^2(\rho^2 - c^2) - (\rho^2 + \rho^2 - c^2)(\rho^2 - a^2) + (\rho^2 - a^2)^2,$$
$$a'^2(a'^2 - c^2) = \rho^2(\rho^2 - c^2) - (\rho^2 + \rho^2 - c^2)(\rho^2 - a'^2) + (\rho^2 - a'^2)^2,$$

we shall have
$$A^2 + B^2 = \frac{(1 + m^2 + m'^2).(\rho^2 - a).(\rho^2 - a'^2)}{\rho^2(\rho^2 - b^2)(\rho^2 - c^2)\left\{\frac{1}{\rho^2} + \frac{m^2}{\rho^2 - b^2} + \frac{m'^2}{\rho^2 - c^2}\right\}^2}.$$

Hence, if R be the semidiameter of the ellipsoid parallel to the chord under consideration,

$$A^2 + B^2 = \frac{R^4 \cos^2\phi\,(\rho^2 - a^2)(\rho^2 - a'^2)}{\rho^2(\rho^2 - b^2)(\rho^2 - c^2)},\quad A = R^2 \cos\phi \left\{\frac{n\cos\psi}{\rho^2 - b^2} + \frac{n'\cos\omega}{\rho^2 - c^2}\right\}.$$

Let us write then $x = \lambda \pm \varpi$, $y = \lambda' \pm \varpi'$, and $z = \lambda'' \pm \varpi''$, and it is easy to see that we shall have

$$\lambda\ = R^2 \cos\phi \left\{\frac{n\cos\psi}{\rho^2 - b^2} + \frac{n'\cos\omega}{\rho^2 - c^2}\right\},\quad \varpi\ = \pm R^2 \cos\phi\, \frac{\sqrt{(\rho^2 - a^2)}.\sqrt{(\rho^2 - a'^2)}}{\rho\sqrt{(\rho^2 - b^2)}.\sqrt{(\rho^2 - c^2)}},$$

$$\lambda' = R^2 \cos\psi \left\{\frac{n_,\cos\phi}{\rho^2} + \frac{n_,'\cos\omega}{\rho^2 - c^2}\right\},\quad \varpi' = \pm R^2 \cos\psi\, \frac{\sqrt{(\rho^2 - a^2)}.\sqrt{(\rho^2 - a'^2)}}{\rho\sqrt{(\rho^2 - b^2)}.\sqrt{(\rho^2 - c^2)}},$$

$$\lambda'' = R^2 \cos\omega \left\{\frac{n_{,,}\cos\phi}{\rho^2} + \frac{n_{,,}'\cos\psi}{\rho^2 - b^2}\right\},\quad \varpi'' = \pm R^2 \cos\omega\, \frac{\sqrt{(\rho^2 - a^2)}.\sqrt{(\rho^2 - a'^2)}}{\rho\sqrt{(\rho^2 - b^2)}.\sqrt{(\rho^2 - c^2)}}.$$

Where n, n', $n_,$, $n_,'$, $n_{,,}$, &c. are connected by the following relations:

$$n_, \cos\psi + n\cos\phi = 0,\quad n_{,,}\cos\omega + n'\cos\phi = 0,\quad n_,'\cos\psi + n_{,,}'\cos\omega = 0.$$

Hence we obtain

$$(a)\quad \frac{\lambda\cos\phi}{\rho^2} + \frac{\lambda'\cos\psi}{\rho^2 - b^2} + \frac{\lambda''\cos\omega}{\rho^2 - c^2} = 0,\quad \varpi^2 + \varpi'^2 + \varpi^2 = \frac{R^4(\rho^2 - a^2)(\rho^2 - a'^2)}{\rho^2(\rho^2 - b^2)(\rho^2 - c^2)}.$$

We can also readily deduce

$$(b)\quad \frac{\lambda^2}{\rho^2}+\frac{\lambda'^2}{\rho^2-b^2}+\frac{\lambda''^2}{\rho^2-c^2}=1-\frac{R^2(\rho^2-a^2).(\rho^2-a'^2)}{\rho^2(\rho^2-b^2).(\rho^2-c^2)},\quad \frac{\varpi\cos\phi}{\rho^2}+\frac{\varpi'\cos\psi}{\rho^2-b^2}+\frac{\varpi''\cos\omega}{\rho^2-c^2}=\frac{\sqrt{(\rho^2-a^2)}.\sqrt{(\rho^2-a'^2)}}{\rho\sqrt{(\rho^2-b^2)}.\sqrt{(\rho^2-c^2)}},$$

$$(c)\quad \varpi\cos\phi+\varpi'\cos\psi+\varpi''\cos\omega=\frac{R^2\sqrt{(\rho^2-a^2)}.\sqrt{(\rho^2-a'^2)}}{\rho\sqrt{(\rho^2-b^2)}.\sqrt{(\rho^2-c^2)}},\quad \frac{\varpi^2}{\rho^2}+\frac{\varpi'^2}{\rho^2-b^2}+\frac{\varpi''^2}{\rho^2-c^2}=\frac{R^2(\rho^2-a^2)(\rho^2-a'^2)}{\rho^2(\rho^2-b^2)(\rho^2-c^2)},$$

$$(d)\quad \frac{\varpi^2}{\rho^4}+\frac{\varpi'^2}{(\rho^2-b^2)^2}+\frac{\varpi''^2}{(\rho^2-c^2)^2}=\frac{R^2(\rho^2-a^2)(\rho^2-a'^2)}{P^2\rho^2(\rho^2-b^2)(\rho^2-c^2)},$$

where P is the perpendicular from the centre upon the tangent plane to the ellipsoid at the point in which the radius vector R pierces the surface.

3. Let (x_0, y_0, z_0, x_1, &c.) be the coordinates of the points in which the chord of the preceding section pierces the surface of the ellipsoid, then

$$\lambda=\frac{x_0+x_1}{2},\quad \lambda'=\frac{y_0+y_1}{2},\quad \lambda''=\frac{z_0+z_1}{2},\quad \varpi=\frac{x_0-x_1}{2},\ \&c.$$

Substitute these values in the preceding equations, and we obtain a number of theorems not devoid of interest or utility. We will proceed to the consideration of the most important. If we call L the length of the chord touching the two confocal surfaces and terminated on either side by the surface of the ellipsoid, the second equation of (a) gives

$$L^2=\frac{4R^4(\rho^2-a^2).(\rho^2-a'^2)}{\rho^2(\rho^2-b^2).(\rho^2-c^2)}.$$

Hence the length of the chord is proportional to the square of the parallel diameter of the ellipsoid, the square of the diameter being equal to the rectangle under the chord and twice a certain right

line (whose geometrical signification we shall presently determine), the square of which is determined by the formula

$$\frac{\rho^2(\rho^2-b^2).(\rho^2-c^2)}{(\rho^2-a^2).(\rho^2-a'^2)}.$$

These expressions were first published, without proof, by MacCullagh. From the value of the chord (L) we obviously obtain the following useful *Theorems*: "In any surface of the second order let three rectangular chords be drawn, common tangents each to any two surfaces confocal with it, the sum of the reciprocals of the three chords is constant." Again, "Let the preceding chords be drawn parallel to any three conjugate semi-diameters, the sum of the three chords is constant."

From the combination of the first equation of (a) and the second of (b), we obtain

$$\frac{x_0\cos\phi}{\rho^2}+\frac{y_0\cos\psi}{\rho^2-b^2}+\frac{z_0\cos\omega}{\rho^2-c^2}=\frac{\sqrt{(\rho^2-a^2)}.\sqrt{(\rho^2-a'^2)}}{\rho\sqrt{(\rho^2-b^2)}.\sqrt{(\rho^2-c^2)}},$$

which gives the following *Theorem:* The sum of the projections upon any right line, drawn from a point upon the surface of an ellipsoid common tangent to two surfaces confocal with it, of the three coordinates of that point, divided each by the square of the semi-major axis of the surface to which it is parallel, is equal to a certain right line, determined by the formula

$$\frac{\sqrt{(\rho^2-a^2)}.\sqrt{(\rho^2-a'^2)}}{\rho\sqrt{(\rho^2-b^2)}.\sqrt{(\rho^2-c^2)}},$$

where a and a' are the semi-major axes of the confocal surfaces. This right line is manifestly the reciprocal of that determined in the last proposition. It is perhaps needless to remark that all these theorems may with greater generality be stated of any central surface of the second order. From the second equation of (c) we perceive that if we write L_1^2 for the sum of the squares of the projections of the common tangent chord L upon the principal axes of the surface, divided each by the square of the semi-major axis of projection, it will be proportional to the square of the parallel diameter of the surface. The square of the diameter being equal to L_1^2 multiplied by the square of the right line

$$\frac{\rho\sqrt{(\rho^2-b^2)}.\sqrt{(\rho^2-c^2)}}{\sqrt{(\rho^2-a^2)}.\sqrt{(\rho^2-a')}}.$$

4. Hitherto we have treated the subject in its utmost generality: to bring it, however, into immediate connexion with MacCullagh's modular theory, we must consider the

two confocal surfaces as degenerated into the two focal curves of the ellipsoid. The respective equations of which, the ellipse and hyperbola, are

$$\frac{x^2}{c^2} + \frac{y^2}{c^2 - b^2} = 1, \quad \frac{x^2}{b^2} + \frac{z^2}{b^2 - c^2} = 1.$$

Let $x_{,}$, $y_{,}$ be the coordinates of a point upon the focal ellipse; x', z' of a point upon the focal hyperbola: then the equations of the right line which passes through these points and makes the angles ϕ, ψ, ω with the principal axes, are

$$y = m(x - x'), \quad z = m'(x - x_1).$$

Let $(x_0, y_0, z_0, x_1$, &c.) be the coordinates of the points in which the right line pierces the surface of the ellipsoid; then if

$$x_0 = \lambda + \varpi, \quad y_0 = \lambda' + \varpi', \quad z_0 = \lambda'' + \varpi'',$$

we shall have

$$\lambda = R^2 \left\{\frac{\cos^2 \psi . x'}{\rho^2 - b^2} + \frac{\cos^2 \omega . x_1}{\rho^2 - c^2}\right\}, \qquad \varpi = \frac{R^2}{\rho} \cos \phi,$$

$$\lambda' = - R^2 \cos \psi \left\{\frac{\cos \phi . x'}{\rho^2} + \frac{\cos \omega . z'}{\rho^2 - c^2}\right\}, \quad \varpi' = \frac{R^2}{\rho} \cos \psi,$$

$$\lambda'' = - R^2 \cos \omega \left\{\frac{\cos \phi . x_1}{\rho^2} + \frac{\cos \psi . y_1}{\rho^2 - b^2}\right\}, \quad \varpi'' = \frac{R^2}{\rho} \cos \omega,$$

where $y_1 \cos \omega + z' \cos \psi = 0$. Hence we can deduce, as in section (2), the following equations:

$$(a') \quad \frac{\lambda \cos \phi}{\rho^2} + \frac{\lambda' \cos \psi}{\rho^2 - b^2} + \frac{\lambda'' \cos \omega}{\rho^2 - c^2} = 0, \quad \varpi^2 + \varpi'^2 + \varpi''^2 = \frac{R^4}{\rho^2},$$

$$(b') \quad \frac{\lambda^2}{\rho^2} + \frac{\lambda'^2}{\rho^2 - b^2} + \frac{\lambda''^2}{\rho^2 - c^2} = 1 - \frac{R^2}{\rho^2}$$

$$\frac{\varpi \cos \phi}{\rho^2} + \frac{\varpi' \cos \psi}{\rho^2 - b^2} + \frac{\varpi'' \cos \omega}{\rho^2 - c^2} = \frac{1}{\rho},$$

$$(c') \quad \varpi \cos \phi + \varpi' \cos \psi + \varpi'' \cos \omega = \frac{R^2}{\rho}, \quad \frac{\varpi^2}{\rho^2} + \frac{\varpi'^2}{\rho^2 - b^2} + \frac{\varpi''^2}{\rho^2 - c^2} = \frac{R^2}{\rho^2},$$

$$(d') \quad \frac{\varpi^2}{\rho^4} + \frac{\varpi'^2}{(\rho^2 - b^2)^2} + \frac{\varpi''^2}{(\rho^2 - c^2)^2} = \frac{R^2}{P^2 \rho^2},$$

where P denotes, as before, the perpendicular from the centre of the ellipsoid upon the tangent plane at the extremity of the radius vector R. Hence if (l) denote the length of the bifocal chord, from the second equation of (a'), we obtain

$$l^2 = \frac{4 R^4}{\rho^2}.$$

So that the bifocal chord is proportional to the parallel diameter of the ellipsoid; the square of the diameter being equal to the rectangle under the chord and the primary axis. We also have

$$\frac{x_0 \cos\phi}{\rho^2} + \frac{y_0 \cos\psi}{\rho^2 - b^2} + \frac{z_0 \cos\omega}{\rho^2 - c^2} = \frac{1}{\rho}.$$

Consequently the sum of the projections upon any bifocal chord, drawn from a point (x_0, y_0, z_0) upon the surface of an ellipsoid, of the three coordinates of that point (divided each by the square of the principal semiaxis of the surface to which it is parallel), is equal to the reciprocal of the semi-major axis.

Let the equation of the paraboloid be

$$\frac{y^2}{p} + \frac{z^2}{q} = x;$$

then the equations of its focal curves are

$$\frac{y^2}{p-q} = x - \frac{q}{4}, \quad \frac{z^2}{q-p} = x - \frac{p}{4}.$$

By a process exactly similar to that we have employed for the ellipsoid, we get

$$\lambda = R^2 \left\{\frac{\cos^2\phi}{2} + \frac{\cos^2\psi . x'}{p} + \frac{\cos^2\omega . x_1}{q}\right\}, \qquad \varpi = \frac{R^2}{2}\cos\phi,$$

$$\lambda' = -R^2 \cos\psi \left\{\frac{z'\cos\omega}{q} - \frac{\cos\phi}{2}\right\}, \qquad \varpi' = \frac{R^2}{2}\cos\psi,$$

$$\lambda'' = -R^2 \cos\omega \left\{\frac{y_1\cos\psi}{p} - \frac{\cos\phi}{2}\right\}, \qquad \varpi'' = \frac{R^2}{2}\cos\omega,$$

where R is determined by the equation

$$\frac{1}{R^2} = \frac{\cos^2\psi}{p} + \frac{\cos^2\omega}{q},$$

the angles ϕ, ψ, ω being the same as in the case of the ellipsoid.

Let (χ) be the length of the bifocal chord, and since $\varpi^2 + \varpi'^2 + \varpi''^2 = \frac{R^4}{4}$, we shall have

$$\chi = R^2 \text{ and } \frac{1}{\chi} = \frac{\cos^2\psi}{p} + \frac{\cos^2\omega}{q}.$$

5. From the second of the equations (c) we get

$$\frac{(x_0 - x_1)^2}{\rho^2} + \frac{(y_0 - y_1)^2}{\rho^2 - b^2} + \frac{(z_0 - z_1)^2}{\rho^2 - c^2} = \frac{4R^2(\rho^2 - a^2).(\rho^2 - a'^2)}{\rho^2(\rho^2 - b^2).(\rho^2 - c^2)}.$$

Expanding and attending to some obvious reductions, we get

$$\frac{x_0x_1}{\rho^2} + \frac{y_0y_1}{\rho^2 - b^2} + \frac{z_0z_1}{\rho^2 - c^2} = 1 - \frac{2R^2.(\rho^2 - a^2)(\rho^2 - a'^2)}{\rho^2.(\rho^2 - b^2)(\rho^2 - c^2)}.$$

But we have already seen that

$$L = \frac{2R^2.\sqrt{(\rho^2 - a^2)}.\sqrt{(\rho^2 - a'^2)}}{\rho.\sqrt{(\rho^2 - b^2)}.\sqrt{(\rho^2 - c^2)}}$$

is the length of the chord parallel to the semidiameter R, and common tangent to the two confocal surfaces whose semi-major axes are a and a'. Hence we deduce the following *Theorem:* "If $(x_0, y_0, z_0, x_1$, &c.) be the points of a given central surface in which a chord common tangent to two surfaces confocal with it, the semi-major axes of which are a and a', pierces that surface, the length L of this chord will be

$$L = \rho \frac{\sqrt{(\rho^2 - b^2)}.\sqrt{(\rho^2 - c^2)}}{\sqrt{(\rho^2 - a^2)}.\sqrt{(\rho^2 - a'^2)}} \left\{1 - \left(\frac{x_0x_1}{\rho^2} + \frac{y_0y_1}{\rho^2 - b^2} + \frac{z_0z_1}{\rho^2 - c^2}\right)\right\}."$$

Let the confocal surfaces degenerate into the focal curves, and we obtain for the length of the bifocal chord,

$$l = \rho . \left\{1 - \left(\frac{x_0x_1}{\rho^2} + \frac{y_0y_1}{\rho^2 - b^2} + \frac{z_0z_1}{\rho^2 - c^2}\right)\right\}:$$

in the paraboloid this becomes

$$\chi = x_1 + x_0 - 2\left(\frac{y_0y_1}{p} + \frac{z_0z_1}{q}\right).$$

The equation

$$(h) \quad \frac{x_0x_1}{\rho^2} + \frac{y_0y_1}{\rho^2 - b^2} + \frac{z_0z_1}{\rho^2 - c^2} = 1 - \frac{2R^2.(\rho^2 - a^2)(\rho^2 - a'^2)}{\rho^2.(\rho^2 - b^2)(\rho^2 - c^2)},$$

if we consider (R) constant, may be regarded as the equation of a plane parallel to the tangent plane to the ellipsoid at the point x_0, y_0, z_0, and passing through the point x_1, y_1, z_1. Let δ be the perpendicular from the point x_0, y_0, z_0 upon this plane, then

$$\delta = \frac{\frac{2R^2}{\rho^2} \cdot \frac{(\rho^2 - a^2).(\rho^2 - a'^2)}{(\rho^2 - b^2).(\rho^2 - c^2)}}{\left\{\frac{x_0^2}{\rho^4} + \frac{y_0^2}{(\rho^2 - b^2)^2} + \frac{z_0^2}{(\rho^2 - c^2)^2}\right\}^{\frac{1}{2}}}.$$

But if (ξ) be the perpendicular from the centre upon the tangent plane at the point (x_0, y_0, z_0),

$$\delta = \frac{2R^2.(\rho^2 - a^2)(\rho^2 - a'^2)}{\rho^2.(\rho^2 - b^2)(\rho^2 - c^2)} . \xi.$$

Hence remembering the value of the chord (L), we shall have

$$\frac{\delta}{L} = \frac{\xi.\sqrt{(\rho^2 - a^2)}.\sqrt{(\rho^2 - a'^2)}}{\rho.\sqrt{(\rho^2 - b^2)}.\sqrt{(\rho^2 - c'^2)}}.$$

Since the plane (h) is parallel to the tangent plane, δ is normal to the surface; therefore $\dfrac{\delta}{L} = \cos\iota_1$, if we denote by ($\iota_1$) the angle which the chord L makes with the normal to the surface. Now since

$$\xi = \frac{\rho.\sqrt{(\rho^2 - b^2)}.\sqrt{(\rho^2 - c^2)}}{\sqrt{(\rho^2 - a^2)}.\sqrt{(\rho^2 - a'^2)}}.\cos\iota_1,$$

we see that the intercept upon the chord (L), made by the central plane parallel to the tangent plane to the ellipsoid at the point from which the chord is drawn, is equal to

$$\frac{\rho.\sqrt{(\rho^2 - b^2)}.\sqrt{(\rho^2 - c^2)}}{\sqrt{(\rho^2 - a^2)}.\sqrt{(\rho^2 - a'^2)}}.$$

Hence follow the *Theorems*, "that if a tangent plane be applied to a central surface, and from the point of contact a chord be drawn common tangent to two surfaces confocal with it, the intercept made by the tangent plane upon the right line drawn from the centre parallel to the common tangent chord is constant." Again, "Along any line of constant curvature, traced upon a central surface of the second order, the preceding common tangent chord makes a constant angle with the normal to the surface." Let the chord (L) become the bifocal chord (l), then

$$\delta = \frac{\dfrac{2R^2}{\rho^2}}{\left\{\dfrac{x_0^2}{\rho^4} + \dfrac{y_0^2}{(\rho^2 - b^2)^2} + \dfrac{z_0^2}{(\rho^2 - c^2)^2}\right\}^{\frac{1}{2}}} = \frac{2R^2}{\rho^2}.\xi.$$

Since $l = \dfrac{2R^2}{\rho}$, we consequently have $\dfrac{\delta}{l} = \dfrac{\xi}{\rho} = \cos\iota_1$, if by ι_1 we denote the angle which the bifocal chord makes with the normal to the surface. Now since $\xi = \rho.\cos\iota_1$, it follows that the central plane, parallel to any tangent plane of the surface, intercepts upon the bifocal chords drawn from the point of contact lengths equal to the semi-major axis of the surface. This important theorem, together with the analogous theorem for the chord common tangent to two confocal surfaces, was first stated by MacCullagh.

6. Let three confocal surfaces intersect in any point (x_0, y_0, z_0), and from this point of intersection let a chord be drawn common tangent to two other surfaces confocal with them: now if $\iota_{,}, \iota_{,,}, \iota_{,,,}$ be the angles which that chord makes with the normals to the three surfaces at their point of intersection, and ξ, η, ζ the three perpendiculars from the centre upon the tangent planes to the three surfaces at the same point of intersection (x_0, y_0, z_0), then, since the proof which we have given for the ellipsoid applies in every respect to the two hyperboloids, we shall have

$$\cos\iota_{,} = \xi . \frac{\sqrt{(\rho^2 - a^2)}.\sqrt{(\rho^2 - a'^2)}}{\rho.\sqrt{(\rho^2 - b^2)}.\sqrt{(\rho^2 - c^2)}}, \quad \cos\iota_{,,} = \eta . \frac{\sqrt{(\mu^2 - a^2)}.\sqrt{(\mu^2 - a'^2)}}{\mu.\sqrt{(\mu^2 - b^2)}.\sqrt{(\mu^2 - c^2)}},$$

$$\cos\iota_{,,,} = \zeta . \frac{\sqrt{(\nu^2 - a^2)}.\sqrt{(\nu^2 - a'^2)}}{\nu.\sqrt{(\nu^2 - b^2)}.\sqrt{(\nu^2 - c^2)}};$$

where μ and ν are the principal semiaxes of the hyperboloids: but since $\xi.\sqrt{(\rho^2 - \mu^2)}.\sqrt{(\rho^2 - \nu^2)} = \rho.\sqrt{(\rho^2 - b^2)}.\sqrt{(\rho^2 - c^2)}$, where $\sqrt{(\rho^2 - \mu^2)}$ and $\sqrt{(\rho^2 - \nu^2)}$ are the principal semiaxes of the central section of the ellipsoid parallel to the tangent plane at the common point of intersection, we can easily see that

$$\cos\iota_{,} = \frac{\sqrt{(\rho^2 - a^2)}.\sqrt{(\rho^2 - a'^2)}}{\sqrt{(\rho^2 - \mu^2)}.\sqrt{(\rho^2 - \nu^2)}}, \quad \cos\iota_{,,} = \frac{\sqrt{(\mu^2 - a^2)}.\sqrt{(\mu^2 - a'^2)}}{\sqrt{(\mu^2 - \rho^2)}.\sqrt{(\mu^2 - \nu^2)}},$$

$$\cos\iota_{,,,} = \frac{\sqrt{(\nu^2 - a^2)}.\sqrt{(\nu^2 - a'^2)}}{\sqrt{(\nu^2 - \rho^2)}.\sqrt{(\nu^2 - \mu^2)}}.$$

Since these curious and very useful expressions were obtained, I have found that they have been published by M. Liouville in a valuable and elaborate memoir in the twelfth volume of his 'Journal. His method however differs altogether from that of the present article. In the case of the bifocal chord, we shall obviously have

$$\cos\iota_{,} = \frac{\xi}{\rho}, \quad \cos\iota_{,,} = \frac{\eta}{\mu}, \quad \cos\iota_{,,,} = \frac{\zeta}{\nu};$$

or remembering that

$$\xi\sqrt{(\rho^2 - \mu^2)}.\sqrt{(\rho^2 - \nu^2)} = \rho.\sqrt{(\rho^2 - b^2)}.\sqrt{(\rho^2 - c^2)},$$

$$\cos\iota_{,} = \frac{\sqrt{(\rho^2 - b^2)}.\sqrt{(\rho^2 - c^2)}}{\sqrt{(\rho^2 - \mu^2)}.\sqrt{(\rho^2 - \nu^2)}}, \quad \cos\iota_{,,} = \frac{\sqrt{(\mu^2 - b^2)}.\sqrt{(\mu^2 - c^2)}}{\sqrt{(\mu^2 - \rho^2)}.\sqrt{(\mu^2 - \nu^2)}},$$

$$\cos\iota_{,,,} = \frac{\sqrt{(\nu^2 - b^2)}.\sqrt{(\nu^2 - c^2)}}{\sqrt{(\nu^2 - \rho^2)}.\sqrt{(\nu^2 - \mu^2)}}.$$

Let (x', y', z') be any point upon the surface of the ellipsoid, and through this point conceive a plane drawn parallel to the tangent plane at (x_0, y_0, z_0); if $L_{,}$ be the intercept made by this plane upon the bifocal chord drawn from the latter point, from the equation (h) we can readily obtain

$$L_{,} = \rho\left\{1 - \frac{x_0 x'}{\rho^2} + \frac{y_0 y'}{\rho^2 - b^2} + \frac{z_0 z'}{\rho^2 - c^2}\right\}.$$

For if $\delta_{,}$ be the perpendicular from the point (x', y', z') upon the plane (h), modified so as to represent the case of the bifocal chord, and ξ the perpendicular from the centre upon the tangent plane at the point (x_0, y_0, z_0), we shall have

$$\delta_{,} = \xi\left\{\frac{x_0 x'}{\rho^2} + \frac{y_0 y'}{\rho^2 - b^2} + \frac{z_0 z'}{\rho^2 - c^2} + \frac{2R^2}{\rho^2} - 1\right\};$$

but since $\delta = \xi\left(\frac{2R^2}{\rho^2}\right)$ subtract, and we shall have

$$\delta - \delta_{,} = \xi\left\{1 - \left(\frac{x_0 x'}{\rho^2} + \frac{y_0 y'}{\rho^2 - b^2} + \frac{z_0 z'}{\rho^2 - c^2}\right)\right\};$$

but $L_{,} = (\delta - \delta_{,}) \sec \iota_{,}$, and $\cos \iota_{,} = \dfrac{\xi}{\rho}$, we consequently have

$$L_{,} = \rho\left\{1 - \left(\frac{x_0 x'}{\rho^2} + \frac{y_0 y'}{\rho^2 - b^2} + \frac{z_0 z'}{\rho^2 - c^2}\right)\right\}:$$

it follows therefore that the intercept $L_{,}$ is always proportional to the distance of x', y', z', from the polar plane of (x_0, y_0, z_0).

7. Since $\cos^2 \iota_{,} + \cos^2 \iota_{,,} + \cos^2 \iota_{,,,} = 1$, we have

$$\frac{\xi^2}{\rho^2} + \frac{\eta^2}{\mu^2} + \frac{\zeta^2}{\nu^2} = 1,$$

which is manifestly the equation of an ellipsoid passing through the centre of the original ellipsoid: but if $(\theta, \theta', \theta'', \theta_{,},$ &c.) be the angles which the perpendiculars (ξ, η, ζ), from the centre upon the tangent planes to the three confocal surfaces at their point of intersection (x_0, y_0, z_0), make with the principal axes, we shall have

$$\frac{x_0}{\rho^2} = \frac{\cos\theta}{\xi}, \qquad \frac{x_0}{\mu^2} = \frac{\cos\theta_{,}}{\eta}, \qquad \frac{x_0}{\nu^2} = \frac{\cos\theta_{,,}}{\zeta};$$

$$\frac{y_0}{\rho^2 - b^2} = \frac{\cos\theta'}{\xi}, \quad \frac{y_0}{\mu^2 - b^2} = \frac{\cos\theta_{,}'}{\eta}, \quad \frac{y_0}{\nu^2 - b^2} = \frac{\cos\theta_{,,}'}{\zeta};$$

$$\frac{z_0}{\rho^2 - c^2} = \frac{\cos\theta''}{\xi}, \quad \frac{z_0}{\mu^2 - c^2} = \frac{\cos\theta_{,}''}{\eta}, \quad \frac{z_0}{\nu^2 - c^2} = \frac{\cos\theta_{,,}''}{\zeta}.$$

We have therefore obviously

$$\frac{1}{x_0^2} = \frac{\xi^2}{\rho^4} + \frac{\eta^2}{\mu^4} + \frac{\zeta^2}{\nu^2}, \quad \frac{1}{y_0^2} = \frac{\xi^2}{(\rho^2 - b^2)^2} + \frac{\eta^2}{(\mu^2 - b^2)^2} + \frac{\zeta^2}{(\nu^2 - b^2)^2},$$

$$\frac{1}{z_0^2} = \frac{\xi^2}{(\rho^2 - c^2)^2} + \frac{\eta^2}{(\mu^2 - c^2)^2} + \frac{\zeta^2}{(\nu^2 - c^2)^2}:$$

from which we see that if the normals to the three confocal surfaces be drawn at their point of intersection, and upon the normals we measure portions equal to the semi-major axes (ρ, μ, ν), and consider them as the principal semi-axes of an ellipsoid having its centre at the point (x_0, y_0, z_0). This new ellipsoid will pass through the centre of the original system, and will be tangent to the plane normal to the semi-major axis ρ. Similarly, if we construct with the same point (x_0, y_0, z_0) as centre surfaces with semi-major axes $\sqrt{(\rho^2 - b^2)}$, $\sqrt{(\mu^2 - b^2)}$, $\sqrt{(\nu^2 - b^2)}$, and $\sqrt{(\rho^2 - c^2)}$, $\sqrt{(\mu^2 - c^2)}$, $\sqrt{(\nu^2 - c^2)}$, they will pass through the centre of the original system and be tangents, the first to the plane normal to the mean axis $\sqrt{(\rho^2 - b^2)}$, and the second to the plane normal to the least axis $\sqrt{(\rho^2 - c^2)}$. The confocal system, therefore, whose centre is the point (x_0, y_0, z_0), and which intersects in the centre of the original system, is

$$(\tau)\ \frac{\xi^2}{\rho^2} + \frac{\eta^2}{\mu^2} + \frac{\zeta^2}{\nu^2} = 1, \qquad (\tau_{,})\ \frac{\xi^2}{\rho^2 - b^2} + \frac{\eta^2}{\mu^2 - b^2} + \frac{\zeta^2}{\nu^2 - b^2} = 1,$$

$$(\tau_{,,})\ \frac{\xi^2}{\rho^2 - c^2} + \frac{\eta^2}{\mu^2 - c^2} + \frac{\zeta^2}{\nu^2 - c^2} = 1;$$

the semi-major axes being in the direction of the normals to the three original confocal surfaces at their point of intersection. From the mere inspection of these equations it follows, that the principal semi-axes of the central section of the ellipsoid (τ) parallel to the principal central section of the original ellipsoid, normal to the semi-major axis ρ, are equal to (b) and (c), *i.e.* they are equal except as to sign, to the semi-major axes of the imaginary focal curve in the principal plane of the original ellipsoid just mentioned; consequently the area of this section is always constant wherever the point (x_0, y_0, z_0) be taken on the surface of the original ellipsoid. Analogous theorems manifestly hold for the hyperboloids $(\tau_{,})$ and $(\tau_{,,})$. It is also a well-known theorem, that in any central surface of the second order, the sum of the squares of the projections of any three conjugate semidiameters upon the normal to any given central section, is constant. From the principles which we have enumerated we can now deduce in

their utmost generality the following most interesting and important theorems due to Professor Chasles.

"Suppose that we are given any central surface of the second order (A), and any fixed plane (B). Through any point in the fixed plane describe three surfaces confocal with the given surface (A), and at this point draw their respective normals; upon each normal measure a portion equal to the semi-major axis of its surface, and construct a new ellipsoid with these right lines for its semi-major axes, and with its centre at the assumed point in the plane (B). This ellipsoid will possess the following properties:*

(1). It will pass through the centre of the ellipsoid (A) and will be tangent to the plane normal to the semi-major axis of (A).

(2). Its central section parallel to the principal plane just mentioned will always have a constant area, whatever be the position of the assumed point in the plane of (B).

(3). Let ι, ι', ι'' be the three angles which the normals to the three confocal surfaces at their point of intersection make with the plane of (B), then

$$\rho^2 \sin^2 \iota + \mu^2 \sin^2 \iota' + \nu^2 \sin^2 \iota'' = a^2,$$

where ρ, μ, ν are the three semi-major axes of the three confocal surfaces, and the constant (a) the semi-major axis of that particular confocal surface which touches the plane (B). If we conceive this plane to pass through the normal to the surface whose semi-major axis is ρ, we shall obtain (as Professor Chasles has done) the known equation for the geodesic line upon a central surface of the second order, viz.

$$\mu^2 \sin^2 \iota' + \nu^2 \sin^2 \iota'' = a^2,$$

where $$PD = \frac{\rho.\sqrt{(\rho^2 - b^2)}.\sqrt{(\rho^2 - c^2)}}{\sqrt{(\rho^2 - a^2)}},$$

P being the perpendicular from the centre upon the plane tangent to the surface at any point upon the geodesic line, and D the semidiameter parallel to the geodesic tangent to the surface at the same point. From the principles enumerated in this section, combined with the value for the intercept upon the bifocal chord given in section (5), we can readily obtain a complete solution of Chasles' most important

* The expressions for $\cos \iota_{,}$, $\cos \iota_{,,}$, &c. given in section (6) enable us to perceive that Prof. Chasles's theorem is in reality but a particular case of a theorem far more general; its consideration I must, however, omit for the present.

proposition, "Given in magnitude and position any three conjugate semidiameters of an ellipsoid, construct the surface." This beautiful proposition is in a very essential manner completed by MacCullagh's value for the central intercept upon the bifocal chord to which we have alluded.

8. In section (6) we have obtained

$$\cos^2\iota_{,} = \frac{(\rho^2 - a^2)(\rho^2 - a'^2)}{(\rho^2 - \mu^2)(\rho^2 - \nu^2)}, \quad \cos^2\iota_{,,} = \frac{(\mu^2 - a^2)(\mu^2 - a'^2)}{(\mu^2 - \rho^2)(\mu^2 - \nu^2)},$$

$$\cos^2\iota_{,,,} = \frac{(\nu^2 - a^2)(\nu^2 - a'^2)}{(\nu^2 - \rho^2)(\nu^2 - \mu^2)}.$$

Hence we have

$$\frac{\cos^2\iota_{,}}{\rho^2 - a^2} + \frac{\cos^2\iota_{,,}}{\mu^2 - a^2} + \frac{\cos^2\iota_{,,,}}{\nu^2 - a^2} = 0, \quad \frac{\cos^2\iota_{,}}{\rho^2 - a'^2} + \frac{\cos^2\iota_{,,}}{\mu^2 - a'^2} + \frac{\cos^2\iota_{,,,}}{\nu^2 - a'^2} = 0,$$

which, if we write thus

$$\frac{\xi^2}{\rho^2 - a^2} + \frac{\eta^2}{\mu^2 - a^2} + \frac{\zeta^2}{\nu^2 - a^2} = 0, \quad \frac{\xi^2}{\rho^2 - a'^2} + \frac{\eta^2}{\mu^2 - a'^2} + \frac{\zeta^2}{\nu^2 - a'^2} = 0,$$

evidently denote the two cones, whose intersections give the four right lines which can in general be drawn from a given point, common tangent to two confocal surfaces. From the nature of the proof it is obvious that the axes of these cones are the normals to the three confocal surfaces which intersect in their common vertex; and since two curves of the second degree cannot touch in more than two points, or intersect in more than four, it is easy to see that they are the cones which envelope the two confocal surfaces whose semi-major axes are a and a'. These equations were first demonstrated by Professors Jacobi and MacCullagh. The focal lines of the preceding cones lie in the plane ξ, ζ, and their equation is

$$\frac{\xi^2}{\rho^2 - \mu^2} + \frac{\zeta^2}{\nu^2 - \mu^2} = 0,$$

which shews that they are parallel to the asymptotes of a central section made in the hyperboloid of one sheet by a plane parallel to the plane of (ξ, ζ). The focal lines are therefore the generatrices of that hyperboloid at the point (x_0, y_0, z_0). This was first stated by Professor Jacobi. From the principles stated we can also deduce the equations of the plane of contact of each cone with either confocal surface. Let (ξ_0, η_0, ζ_0) be the coordinates of the common centre of the system of confocal surfaces referred to the three normals at the point (x_0, y_0, z_0). The equation of the plane of contact of the cone circumscribing the surface whose semi-major axis

is α, then becomes

$$\frac{\xi\xi_0}{\rho^2-\alpha^2}+\frac{\eta\eta_0}{\mu^2-\alpha^2}+\frac{\zeta\zeta_0}{\nu^2-\alpha^2}=1.$$

This equation was first given by MacCullagh. Hence Professor Graves has derived his beautiful theorem for the length of any side of the cone of contact, viz.

$$\frac{1}{P^2L^2}=\frac{\cos^2\iota_{,}}{(\rho^2-\alpha^2)^2}+\frac{\cos^2\iota_{,,}}{(\mu^2-\alpha^2)^2}+\frac{\cos^2\iota_{,,,}}{(\nu^2-\alpha^2)^2},$$

where L indicates the distance from the summit of the cone to any point of contact with the ellipsoid, and P the perpendicular from the centre upon the plane tangent to the cone along that side of contact: $\iota_{,}$, $\iota_{,,}$, $\iota_{,,,}$ are the angles which this side makes with the normals to the three confocal surfaces at their point of intersection in the summit of the cone. As a particular case of this more general theorem, Professor Graves obtains Joachimsthal's theorem for the geodesic line and line of curvature, viz. PD = constant. The preceding beautiful theorem, with many others relating to geodesic lines and lines of curvature traced upon confocal surfaces, Professor Graves has recently published in the *Proceedings of the Royal Irish Academy*. Of all these theorems we can, however, with facility deduce independent proofs from the principles stated in the present section.

Let us conceive the side (L) to be any one of the four sides which touch the confocal surface whose semi-major axis is (α'); then attending to the values of ($\cos\iota_{,}$, $\cos\iota_{,,}$, $\cos\iota_{,,,}$) in this case, we can with facility deduce the interesting and (so far as I am aware) original expression

$$P^2L^2=\frac{(\rho^2-\alpha^2)(\mu^2-\alpha^2)(\nu^2-\alpha^2)}{(\alpha^2-\alpha'^2)},$$

from which by differentiation we obtain

$$\frac{dP}{P}+\frac{dL}{L}=\frac{\rho d\rho}{\rho^2-\alpha^2}+\frac{\mu d\mu}{\mu^2-\alpha^2}+\frac{\nu d\nu}{\nu^2-\alpha^2},$$

an expression by no means without utility in the solution of many propositions. Let α' become ρ, and we shall have the curious value

$$PL=(\alpha^2-\mu^2)(\nu^2-\alpha^2),$$

the interpretation of which is obvious.

If ds, ds', ds'' be the elements along the normals to the

three confocal surfaces which pass through the summit of the cone, their known values are

$$ds = \frac{\sqrt{(\rho^2 - \mu^2)}.\sqrt{(\rho^2 - \nu^2)}}{\sqrt{(\rho^2 - b^2)}.\sqrt{(\rho^2 - c^2)}}\,d\rho,\quad ds' = \frac{\sqrt{(\mu^2 - \rho^2)}.\sqrt{(\mu^2 - \nu^2)}}{\sqrt{(\mu^2 - b^2)}.\sqrt{(\mu^2 - c^2)}}\,d\mu,$$

$$ds'' = \frac{\sqrt{(\nu^2 - \rho^2)}.\sqrt{(\nu^2 - \mu^2)}}{\sqrt{(\nu^2 - b^2)}.\sqrt{(\nu^2 - c^2)}}\,d\nu.$$

Now if $d\sigma$ be the element of any side of contact (L) of the cone, its value will evidently be

$$d\sigma = ds.\cos\iota_{,} + ds'.\cos\iota_{,,} + ds''.\cos\iota_{,,,}.$$

Substitute their values for $(ds,\ ds',$ &c. $\cos\iota_{,},$ &c.), and we shall have

$$d\sigma = \frac{\sqrt{(\rho^2 - a^2)}.\sqrt{(\rho^2 - a'^2)}}{\sqrt{(\rho^2 - b^2)}.\sqrt{(\rho^2 - c^2)}}\,d\rho + \frac{\sqrt{(\mu^2 - a^2)}.\sqrt{(\mu^2 - a'^2)}}{\sqrt{(\mu^2 - b^2)}.\sqrt{(\mu^2 - c^2)}}\,d\mu$$
$$+ \frac{\sqrt{(\nu^2 - a^2)}.\sqrt{(\nu^2 - a'^2)}}{\sqrt{(\nu^2 - b^2)}.\sqrt{(\nu^2 - c^2)}}\,d\nu,$$

an expression which in itself includes a number of theorems not devoid of interest or utility, but which we must for the present omit as not being essentially connected with the subject of the present paper. The cones whose intersections give the four bifocal chords, which can in general be drawn from a point upon the surface of an ellipsoid, are

$$\frac{\xi^2}{\rho^2 - b^2} + \frac{\eta^2}{\mu^2 - b^2} + \frac{\zeta^2}{\nu^2 - b^2} = 0,\quad \frac{\xi^2}{\rho^2 - c^2} + \frac{\eta^2}{\mu^2 - c^2} + \frac{\zeta^2}{\nu^2 - c^2} = 0.$$

From the nature of our proof it is manifest that they are the cones which pass through the focal curves; the first through the focal hyperbola, the second through the focal ellipse, their principal axes being the normals to the three confocal surfaces which intersect in their common vertex upon the surface of the ellipsoid. It is at least curious to observe how naturally a few not inelegant expressions lead to the most important theorems of the geometry of surfaces of the second order; they in fact anticipate the theory of elliptic coordinates. This subject, however, I hope to develope more in detail on some future occasion.

9. Through any point (x_0, y_0, z_0) upon the surface of an ellipsoid let two confocal hyperboloids be described, and at the same point draw their respective normals. From this point draw also a bifocal chord, and let it make with the three normals, angles $(\iota_{,},\ \iota_{,,},\ \iota_{,,,})$. Conceive this bifocal chord

to be orthogonally projected upon the plane tangent to the ellipsoid at the point (x_0, y_0, z_0), and let this projected right line make angles (ι', ι'') with the normals to the hyperboloids, then

$$\cos \iota' = \frac{\cos \iota_{,,}}{\sin \iota_{,}}, \quad \cos \iota'' = \frac{\cos \iota_{,,,}}{\sin \iota_{,}}.$$

If μ and ν be the semi-major axes of the hyperboloids, we shall have $\rho^2 - \mu^2$ and $\rho^2 - \nu^2$ equal to the squares of the principal semi-axes of the central section of the ellipsoid parallel to the tangent plane at the point (x_0, y_0, z_0). Let D be the semi-diameter of this section parallel to the projection of the bifocal chord upon the tangent plane, and let ξ, η, ζ be the perpendiculars from the centre upon the planes tangent to the three confocal surfaces at their common point of intersection. Then, remembering the values of $\cos \iota_{,}$, $\cos \iota_{,,}$, $\cos \iota_{,,,}$, we shall have

$$\frac{\cos^2 \iota'}{\rho^2 - \mu^2} + \frac{\cos^2 \iota''}{\rho^2 - \nu^2} = \frac{1}{D^2},$$

$$\frac{\rho^2}{\rho^2 - \xi^2} \left\{ \frac{\eta^2 (\rho^2 - \nu^2)}{\mu^2} + \frac{\zeta^2 (\rho^2 - \mu^2)}{\nu^2} \right\} = \frac{\rho^2 (\rho^2 - b^2).(\rho^2 - c^2)}{\xi^2 D^2}.$$

Hence

$$\mu^2 \sin^2 \iota' + \nu^2 \sin^2 \iota'' = 2\rho^2 - (\rho^2 - \mu^2 + \rho^2 - \nu^2) - \frac{\rho^2}{\rho^2 - \xi^2} (\eta^2 + \zeta^2);$$

and since
$$\frac{\rho^2 - \xi^2}{\rho^2} = \frac{\eta^2}{\mu^2} + \frac{\zeta^2}{\nu^2},$$

we can readily deduce for the geodesic line the known equation

$$\mu^2 \sin^2 \iota' + \nu^2 \sin^2 \iota'' = \rho^2 - \frac{\rho^2 (\rho^2 - b^2).(\rho^2 - c^2)}{\xi^2 D^2} = a^2;$$

we consequently have

$$(k) \quad \rho^2 - a^2 = \rho^2 - b^2 + \rho^2 - c^2 - \frac{\xi^2 (\rho^2 - R^2)}{(\rho^2 - \xi^2)},$$

where R is the semidiameter drawn to the point (x_0, y_0, z_0) upon the surface of the ellipsoid. Now since

$$R^2 = x_0^2 + y_0^2 + z_0^2, \quad \frac{1}{\xi^2} = \frac{x_0^2}{\rho^4} + \frac{y_0^2}{(\rho^2 - b^2)^2} + \frac{z_0^2}{(\rho^2 - c^2)^2},$$

we can express in terms of the coordinates of this particular point (where the osculating plane of the geodesic line contains two bifocal chords of the surface) the semi-major axis (a) of the confocal surface, which the geodesic line touches.

For we must remember that the equation (k) expresses, in terms of the ξ and R of any point, not the a of *any* geodesic line which passes through that point, but only the a of that particular geodesic line whose osculating plane at that point contains a pair of bifocal chords: so that from point to point of the surface we in general obtain a different (a). If, however, we generalize the equation (k) by the substitution of their values for ξ and R, we shall have

$$1=\frac{x^2}{\rho^2}+\frac{y^2+z^2}{\rho^2-b^2+\rho^2-c^2+a^2}+\frac{(a^2-b^2+\rho^2-c^2)\rho^2}{\rho^2-b^2+\rho^2-c^2+a^2}\left\{\frac{y^2}{(\rho^2-b^2)^2}+\frac{z^2}{\rho^2-c^2)^2}\right\},$$

the equation of an ellipsoid whose semi-major axis is that of the original ellipsoid. Let us find its curve of intersection with the original ellipsoid $\frac{x^2}{\rho^2}+\frac{y^2}{\rho^2-b^2}+\frac{z^2}{\rho^2-c^2}=1$, and we shall have

$$\frac{\rho^2-c^2+a^2}{\rho^2-b^2}y^2+\frac{\rho^2-b^2+a^2}{\rho^2-c^2}z^2=(\rho^2-c^2+a^2-b^2)\rho^2\left\{\frac{y^2}{(\rho^2-b^2)^2}+\frac{z^2}{(\rho^2-c^2)^2}\right\},$$

which becomes, after some obvious reductions,

$$\frac{b^2(a^2-c^2)}{(\rho^2-b^2)^2}y^2+\frac{c^2(a^2-b^2)}{(\rho^2-c^2)^2}z^2=0,$$

$$\text{or}\quad (k')\quad \frac{b^2y^2}{(\rho^2-b^2)^2.(a^2-b^2)}+\frac{c^2z^2}{(\rho^2-c^2)^2.(a^2-c^2)}=0,$$

the equation of two right lines in the plane of (yz) so long as a^2 is greater than b^2 and less than c^2. Hence we see that if two planes pass through these right lines and intersect in the semi-major axis of the surface, we shall have two sections analogous to the circular sections, and that the curves which they form upon the surface of the ellipsoid will be the locus of the points, upon the multiplicity of geodesic lines which touch the same confocal surface (whose semi-major axis is a), at which the osculating planes contain pairs of bifocal chords. This conclusion we believe to be given for the first time. The different particular cases, the limits of our paper oblige us for the present to omit.

From the equation (k') we obtain

$$\frac{(\rho^2-c^2)^2}{(\rho^2-b^2)^2}\cdot\frac{y^2}{z^2}=\frac{c^2}{b^2}\cdot\frac{a^2-b^2}{c^2-a^2}=T^2.$$

if by (T) we denote the quantity $\frac{(\rho^2-c^2)}{(\rho^2-b^2)}\cdot\frac{y}{z}$. Let the y and z of the last equation indicate the points in which either

plane of the equation (k') intersects the elliptic curve in the principal section (yz), the quantity (T) will then denote the tangent of the angle which the right line, tangent to the principal section at this point, makes with the axis of (y). If (σ) indicate this angle, we shall have

$$T^2 = \tan^2\sigma = \frac{c^2}{b^2}\cdot\frac{a^2-b^2}{c^2-a^2};$$

consequently, for any given (σ) we can determine (a), and *vice versâ*. From the last equation we obtain

$$a^2 = \frac{(1+T^2)\,b^2c^2}{c^2+b^2T^2} = \frac{b^2c^2}{c^2\cos^2\sigma + b^2\sin^2\sigma}.$$

Now if in the principal plane (yz) we construct the ellipse $\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, which is, in fact, the conjugate of the focal curve of the ellipsoid in this plane, since

$$\frac{1}{r^2} = \frac{\cos^2\sigma}{b^2} + \frac{\sin^2\sigma}{c^2},$$

we shall have the very curious value $a^2 = r^2$. The maximum and minimum values of (a) are therefore (c) and (b), and the confocal surface, of which (a) is the semi-major axis, must in general be a hyperboloid of one sheet. The preceding investigation is obviously but a particular case of a still more general and, perhaps, more interesting investigation, viz. where the osculating plane contains a pair of right lines common tangents to any two confocal surfaces. The materials for the solution of this more general theorem we have furnished in section (6), but farther we cannot proceed at present. If we project the right lines (k') upon the circular sections, we shall have as their equations

$$\frac{y^2}{(\rho^2-b^2).(a^2-b^2)} + \frac{z^2}{(\rho^2-c^2).(a^2-c^2)} = 0.$$

10. We will now as briefly as possible state the modifications which some of the preceding theorems undergo in the case of the paraboloids, the verification of which the reader can readily supply for himself.

Let two confocal paraboloids intersect, and let normals be applied to them at any point (P) of their intersection. Draw a bifocal chord of each surface parallel to the normal of the other, the two chords so drawn will be equal and of a constant length, wherever the point P is taken in the line of intersection of the surfaces; that constant length being equal

to the difference between the parameters of either pair of coincident principal sections.

Let $\frac{y^2}{p} + \frac{z^2}{q} = x$ be the equation of a paraboloid, and let a chord which pierces its surface in the points $(x_0, y_0, z_0, x_1, \&c.)$ be drawn common tangent to two confocal paraboloids, of which the equations are

$$\frac{y^2}{p_0} + \frac{z^2}{q_0} = x + h_0, \quad \frac{y^2}{p_1} + \frac{z^2}{q_1} = x + h_1,$$

where $p_0 - p = 4h_0$ and $p_1 - p = 4h_1$; then, if $\varpi = \frac{1}{2}(x_0 - x_1)$, $\varpi' = \frac{1}{2}(y_0 - y_1)$, and $\varpi'' = \frac{1}{2}(z_0 - z_1)$, we shall have

$$\varpi = \frac{\chi}{2} \frac{\sqrt{(p - p_0)}.\sqrt{(p - p_1)}}{\sqrt{(pq)}} \cos\phi, \quad \varpi' = \frac{\chi}{2} \frac{\sqrt{(p - p_0)}.\sqrt{(p - p_1)}}{\sqrt{(pq)}} \cos\psi,$$

$$\varpi'' = \frac{\chi}{2} \frac{\sqrt{(p - p_0)}.\sqrt{(p - p_1)}}{\sqrt{(pq)}} \cos\omega.$$

where χ is the bifocal chord of the paraboloid parallel to the common tangent chord, and ϕ, ψ, ω the angles which it makes with the principal axes. If (L) be the length of the common tangent chord, we shall have

$$L = \chi \frac{\sqrt{(p - p_0)}.\sqrt{(p - p_1)}}{\sqrt{(pq)}}.$$

We have already stated the values for these expressions in the particular case in which the confocal paraboloids degenerate into the focal curves of the original paraboloid (vide section IV.). Let $(\iota_{,}, \iota_{,,}, \iota_{,,,})$ be the angles which the chord (L) makes with the normals to the three confocal paraboloids which intersect in the point (x_0, y_0, z_0), then

$$\cos\iota_{,} = \frac{\sqrt{(p - p_0)}.\sqrt{(p - p_1)}}{\sqrt{(\chi_1 \chi')}}, \quad \cos\iota_{,,} = \frac{\sqrt{(p' - p_0)}.\sqrt{(p' - p_0)}}{\sqrt{\chi_1}.\sqrt{(\chi_1 - \chi')}},$$

$$\cos\iota_{,,,} = \frac{\sqrt{(p'' - p_0)}.\sqrt{(p'' - p_1)}}{\sqrt{\chi'}.\sqrt{(\chi' - \chi_1)}},$$

where p' and p'' correspond to p in the original paraboloid and $\chi_1 = p' - p$ and $\chi' = p'' - p$. Hence we obtain as the equations of the cones, the intersection of which gives the four right lines which can in general be drawn from a given point common tangent to two confocal paraboloids,

$$\frac{\xi^2}{p - p_0} + \frac{\eta}{p' - p_0} + \frac{\zeta^2}{p'' - p_0} = 0, \quad \frac{\xi^2}{p - p_1} + \frac{\eta^2}{p' - p_1} + \frac{\zeta^2}{p'' - p_1} = 0.$$

The axes of these cones are manifestly the normals to the three confocal paraboloids which intersect in their common vertex. It is easy to see that the first is the equation of the cone, which, having its vertex at the point (x_0, y_0, z_0) in which the three confocal paraboloids intersect, circumscribes the paraboloid whose equation is

$$\frac{y^2}{p_0} + \frac{z^2}{q_0} = x + h_0.$$

Similarly for the second: and since the equation of their focal lines is

$$\frac{\xi^2}{p' - p} + \frac{\zeta^2}{p' - p''} = 0,$$

it follows that their focal lines are the generatrices of the hyperbolic paraboloid at the point (x_0, y_0, z_0). If we conceive the chord (L) to pass through the focal curves, the preceding equations become

$$\cos \iota_{,} = \frac{\sqrt{(pq)}}{\sqrt{(\chi_1 \chi')}}, \quad \cos \iota_{,,} = \frac{\sqrt{(p'q')}}{\sqrt{\chi_1} \cdot \sqrt{(\chi_1 - \chi')}},$$

$$\cos \iota_{,,,} = \frac{\sqrt{(p''q'')}}{\sqrt{\chi'} \cdot \sqrt{(\chi' - \chi_1)}},$$

which we can easily shew to be the same as

$$\cos \iota' = \frac{\xi}{x_0}, \quad \cos \iota_{,,} = \frac{\eta}{x_0 + 2h'}, \quad \cos \iota_{,,,} = \frac{\zeta}{x_0 + 2h''},$$

where ξ, η, ζ denote the perpendiculars from the summit of the original paraboloid upon the planes tangent to the three confocal paraboloids at their point of intersection (x_0, y_0, z_0) and $p' - p = 4h'$ and $p'' - p = 4h''$. Since $x_0 \cos \iota_{,} = \xi$, we have the following theorem: If at any point (x_0, y_0, z_0) upon the surface of a paraboloid a tangent plane be applied, and from the same point a bifocal chord be drawn, the intercept made by this plane upon the right line drawn from the summit of the paraboloid parallel to the bifocal chord, will be equal to the coordinate (x_0) of the point from which the bifocal chord is drawn. The equations of the cones whose intersection gives us the four bifocal chords which can, in general, be drawn from a point upon the surface of a paraboloid, are

$$\frac{\xi^2}{p} + \frac{\eta^2}{p'} + \frac{\zeta^2}{p''} = 0, \quad \frac{\xi^2}{q} + \frac{\eta^2}{q'} + \frac{\zeta^2}{q''} = 0.$$

They are manifestly the equations of two cones which have

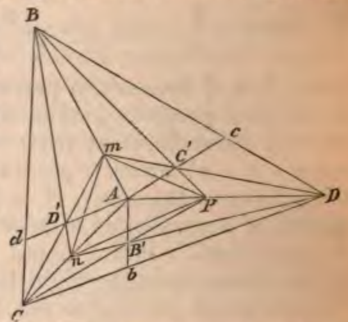
a common vertex at the point (x_0, y_0, z_0), the first of them passing through the focal curve in the plane of (xz), the second through the focal curve in the plane of (xy). We believe that several of the theorems in the present section are given for the first time: however, be that as it may, attention to the principles we have indicated will enable us, any proposition being stated for the ellipsoid, to state, in general, an analogous proposition for the paraboloid.

ANALYTICAL DEMONSTRATION OF A THEOREM IN CARNOT'S 'THÉORIE DES TRANSVERSALES.'

By William Walton.

The final theorem in that branch of the *Essai sur la Théorie des Transversales*, in which Carnot treats of the intersection of straight lines, consists of three propositions of which, in accordance with the system of his essay, he has given elegant geometrical demonstrations. I am not aware that this theorem has been demonstrated analytically, and have therefore thought that the following proof might be acceptable to the readers of the *Journal*. The theorem to which I allude is the following:

"*Si sur chacune des arêtes qui partent du sommet A d'une pyramide triangulaire $ABCD$, on prend à volonté un point m, n, p, pour former le triangle mnp sur la surface extérieure de cette pyramide, et qu'ayant imaginé les diagonales $\overline{Bn}, \overline{Cm}, \overline{Cp}, \overline{Dn}, \overline{Dm}, \overline{Bp}$, on mène encore par le sommet A et les points de croisement D', B', C', les transversales $\overline{Ad}, \overline{Ab}, \overline{Ac}$. Je dis que,*



1°. *Les transversales $\overline{Dd}, \overline{Bb}, \overline{Cc}$, se croiseront toutes en un même point A' de la base.*

2°. *Les quatre transversales $\overline{AA'}, \overline{BB'}, \overline{CC'}, \overline{DD'}$, se croiseront aussi toutes en un même point K de l'espace.*

3°. *Le plan qui passera par les trois points B', C', D', et les deux autres plans BCD, mnp, auront tous trois une intersection commune.*"

Before proceeding to the analytical demonstration of this theorem, I will premise two propositions, which are particularly useful in various problems concerning rectilinear transversals, and which will form the basis of the investigations in this paper.

1. To prove that any three lines, which intersect in one point, may be represented by the three equations

$$v - w = 0, \quad w - u = 0, \quad u - v = 0.$$

Let the equations to any two of the lines be

$$p = 0, \quad q = 0:$$

that to the third will be $p + kq = 0$,

k being some constant quantity.

Hence, any three lines whatever, which cut each other in one point, may be represented by the three equations

$$p = 0, \quad kq = 0, \quad -p - kq = 0,$$

or, putting $p = v - w$, and $kq = w - u$, by the three equations

$$v - w = 0, \quad w - u = 0, \quad u - v = 0.$$

2. Three straight lines A, B, C, pass through a single point: X, Y, Z, are three other straight lines such that Y, Z, intersect in A; Z, X, in B; and X, Y, in C. To prove that the lines A, B, C, X, Y, Z, may be represented by the following system of equations:

A	B	C
$v - w = 0$	$w - u = 0$	$u - v = 0$
X	Y	Z
$v + w = \lambda$	$w + u = \lambda$	$u + v = \lambda$

The first three equations, as we know by proposition (1), will always represent A, B, C.

Again, since Y, Z, intersect in A, the equations to Y, Z, must be of the forms

$$p = 0, \quad ap + v - w = 0,$$

where a is a constant; or, putting $ap = \lambda - u - v$, of the forms

$$u + v = \lambda, \quad w + u = \lambda.$$

Again, it is obvious that a line, of which the equation is

$$v + w = \lambda,$$

intersects Y in C, and Z in B: this line must therefore necessarily be X.

3. In the following investigation I shall assume the obvious proposition that any number of straight lines will intersect in space, provided that they intersect when projected upon an arbitrary plane, and shall therefore confine my attention to the equations to the projections of the straight lines of the diagram upon an arbitrary coordinate plane.

Let the equations to AB, AC, AD, be

AB	AC	AD
$v - w = 0$	$w - u = 0$	$u - v = 0$

Then the equations to nD, DB, Bn; pB, BC, Cp; mC, CD, Dm: will be

nD	DB	Bn
$v + w = \lambda$	$w + u = \lambda$	$u + v = \lambda$

Cp	pB	BC
$v + w = \mu$	$w + u = \mu$	$u + v = \mu$

CD	Dm	mC
$v + w = \nu$	$w + u = \nu$	$u + v = \nu$

But CD, DB, BC, must have equations of the form

CD	DB	BC
$v + w = \rho$	$w + u = \rho$	$u + v = \rho$

In order that the two equations to CD given above may be equivalent to each other, we must have, a being some constant,

$$(1 - a)(v + w - \rho) = v + w - \nu,$$

or

$$\nu = a(v + w) + (1 - a)\rho.$$

Similarly, β, γ, being constants,

$$\lambda = \beta(w + u) + (1 - \beta)\rho,$$

$$\mu = \gamma(u + v) + (1 - \gamma)\rho.$$

Hence the equations to mC, Dm, nD, Bn, pB, Cp become,

$$mC \ldots\ldots u + (1 - a)v - aw = (1 - a)\rho,$$

$$Dm \ldots\ldots u - av + (1 - a)w = (1 - a)\rho,$$

$$nD \ldots\ldots v + (1 - \beta)w - \beta u = (1 - \beta)\rho,$$

$$Bn \ldots\ldots v - \beta w + (1 - \beta)u = (1 - \beta)\rho,$$

$$pB \ldots\ldots w + (1 - \gamma)u - \gamma v = (1 - \gamma)\rho,$$

$$Cp \ldots\ldots w - \gamma u + (1 - \gamma)v = (1 - \gamma)\rho.$$

The equation to Ab, which passes through the intersection of AB, AC, AD, and through that of nD, Cp, will be

$$Ab. \ldots\ldots (\beta - \gamma)\,u - \beta(1-\gamma)\,v + \gamma(1-\beta)\,w = 0.$$

The equation to Bb, which passes through the intersection of DB, BC, and of CD, Ab, is

$$Bb. \ldots (\beta - \gamma)\,u - \gamma(1-\beta)\,v + \beta(1-\gamma)\,w = (\beta-\gamma)\,\rho.$$

By symmetry we see that the equations to Cc, Dd, will be

$$Cc. \ldots (\gamma - \alpha)\,v - \alpha(1-\gamma)\,w + \gamma(1-\alpha)\,u = (\gamma-\alpha)\,\rho,$$

$$Dd. \ldots (\alpha - \beta)\,w - \beta(1-\alpha)\,u + \alpha(1-\beta)\,v = (\alpha-\beta)\,\rho.$$

The three lines Bb, Cc, Dd, all intersect at a point in a line of which the equation is

$$\alpha(\beta-\gamma)\,u + \beta(\gamma-\alpha)\,v + \gamma(\alpha-\beta)\,w = 0.$$

4. The equation to AA', which passes through the intersection of AB, AC, AD, and of Bb, Cc, Dd, is

$$AA'. \ldots \alpha(\beta-\gamma)\,u + \beta(\gamma-\alpha)\,v + \gamma(\alpha-\beta)\,w = 0,$$

being the equation just obtained.

The equation to BB', which passes through the intersection of DB, BC, and of nD, Cp, is

$$BB'. \ldots (\beta-\gamma)\,u - \gamma v + \beta w = (\beta-\gamma)\,\rho.$$

By symmetry we know that the equations to CC', DD', will be

$$CC'. \ldots (\gamma-\alpha)\,v - \alpha w + \gamma u = (\gamma-\alpha)\,\rho,$$

$$DD'. \ldots (\alpha-\beta)\,w - \beta u + \alpha v = (\alpha-\beta)\,\rho.$$

If we add together the equations to BB', CC', DD', we obtain an identical equation: these three lines therefore intersect in a single point. Again, if we multiply them in order by α, β, γ, and then add, we arrive at an equation coinciding with that to AA', which shews that their point of intersection lies in AA'.

5. Again, the equation to mn, which passes through the intersection of Cm, AB, and of Bn, AC, is

$$mn. \ldots \frac{1}{1-\alpha}\,u + \frac{1}{1-\beta}\,v - \left(\frac{\alpha}{1-\alpha} + \frac{\beta}{1-\beta}\right) w = \rho.$$

Similarly, the equation to np is

$$np. \ldots \frac{1}{1-\beta}\,v + \frac{1}{1-\gamma}\,w - \left(\frac{\beta}{1-\beta} + \frac{\gamma}{1-\gamma}\right) u = \rho.$$

The equation to a line through the intersection of *mn*, *BC*, and of *np*, *CD*, is

$$\left(\frac{\beta}{1-\beta}+\frac{\gamma}{1-\gamma}\right)u+\left(\frac{\gamma}{1-\gamma}+\frac{\alpha}{1-\alpha}\right)v$$
$$+\left(\frac{\alpha}{1-\alpha}+\frac{\beta}{1-\beta}\right)w=\left(\frac{\alpha}{1-\alpha}+\frac{\beta}{1-\beta}+\frac{\gamma}{1-\gamma}\right)\rho.$$

Symmetry shews that this is the equation also to a line passing through the intersection of *np*, *CD*, and of *pm*, *DB*: this is therefore the equation to a line in which the three intersections lie. It is therefore the equation to the intersection of the two planes *BCD*, *mnp*.

Again, the equation to *bd*, which passes through the intersection of *Ab*, *CD*, and of *Ad*, *BC*, is

$$bd\ldots\left(\frac{\beta}{1-\beta}-\frac{\gamma}{1-\gamma}\right)u-\left(\frac{\alpha}{1-\alpha}+\frac{\gamma}{1-\gamma}\right)v$$
$$-\left(\frac{\alpha}{1-\alpha}-\frac{\beta}{1-\beta}\right)w+\rho\left(\frac{\gamma}{1-\gamma}+\frac{\alpha}{1-\alpha}-\frac{\beta}{1-\beta}\right)=0.$$

The equation to $B'D'$, which passes through the intersection of *Bn*, *Cm*, and of *Cp*, *Dn*, is

$$B'D'\ldots\left(\frac{\beta}{1-\beta}-\frac{\gamma}{1-\gamma}+\frac{\alpha}{1-\alpha}\cdot\frac{\beta}{1-\beta}\right)u-\left(\frac{\alpha}{1-\alpha}+\frac{\gamma}{1-\gamma}\right)\left(1+\frac{\beta}{1-\beta}\right)v$$
$$+\left(\frac{\beta}{1-\beta}-\frac{\alpha}{1-\alpha}+\frac{\gamma}{1-\gamma}\cdot\frac{\beta}{1-\beta}\right)w+\rho\left(\frac{\alpha}{1-\alpha}+\frac{\gamma}{1-\gamma}-\frac{\beta}{1-\beta}\right)=0.$$

At the intersection of $B'D'$ and *bd*, we obtain, from their equations,

$$\rho=u+w,$$
$$\frac{\alpha}{1-\alpha}u-\left(\frac{\alpha}{1-\alpha}+\frac{\gamma}{1-\gamma}\right)v+\frac{\gamma}{1-\gamma}w=0.$$

But, if we put $u+w$ for ρ in the equation to the line of intersection of the planes *BCD*, *mnp*, we obtain this last equation. Hence $B'D'$, *bd*, intersect in this line. Similarly, by virtue of symmetry, $C'B'$, *cb*; $D'C'$, *dc*; also intersect in it. The intersections therefore of the three planes *BCD*, *mnp*, $B'C'D'$, coincide.

December 6, 1849.

ON THE MEANING OF THE EQUATION $U^2 = V^2$, WHEN U AND V ARE PRODUCTS OF n LINEAR FUNCTIONS OF TWO VARIABLES.

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1. From the equation

$$U = x^2P - y(y + ax)Q = 0$$

comes, when $x = 0 = y$, if these values do not cause either P or Q to vanish,

$$\left(\frac{dy}{dx}\right)^2 + a\frac{dy}{dx} = \frac{P}{Q},$$

giving $$y_1' + y_2' + a = 0,$$

y_1' and y_2' being the roots, supposed possible, of the quadratic.

If in the equation to any transversal

$$\frac{1}{x} - \alpha\frac{y}{x} - \beta = 0,$$

we put successively for y the values $y = 0$, $y = -ax$, $y = y_1'x$, $y = -(y_1' + a)x$, we obtain

$$\frac{1}{x_1} + \frac{1}{x_2} = \frac{1}{x_3} + \frac{1}{x_4}.$$

Hence we conclude that, if $p = 0$, $q = 0$, $r = 0$ be the equations of any three lines meeting in a point, for the coordinates of which $P:Q$ has a positive and determinate value, the equation $$U = p^2P - qrQ = 0$$

represents a curve having two branches through that point, such that, if from any point in the line $p = 0$ a transversal cut the pair $q = 0$, $r = 0$, and the pair of tangents to the curve at the intersection of the three lines, the segments of it intercepted by the pairs $q = 0$ and $r = 0$ have a common harmonic mean with those intercepted by the two tangents. If $r = q$, or $$p^2P - q^2Q = 0 = R,$$

be the curve, the tangents at the intersection of $p = 0$ and $q = 0$ form with them an harmonic pencil, the two latter being a corresponding pair.

Let this be denoted by saying that the two branches of $R = 0$ are *synharmonicals* in respect of $p = 0$ and $q = 0$.

2. If $u_1 = 0$, $v_1 = 0$; $u_2 = 0$, $v_2 = 0$; .... be the equations to $2n$ right lines meeting two and two $(u_s v_s)$ in n points (ϖ)

of a right line $A = 0$, and produced to intersect in certain points (p) not in A, no three being supposed to meet in a point (p), and if U_n denote a function of x and y of the n^{th} degree,

$$U_n = u_1 u_2 \ldots . u_n - \lambda v_1 v_2 \ldots . v_n = 0$$

represents, if λ be properly determined, a curve of the n^{th} order that contains $(n+1)$, and consequently all, points of $A = 0$; or

$$U_n = U_{n-1} A = 0;$$

where U_{n-1} is a curve of the $(n-1)^{\text{th}}$ order, on which lie all the intersections of any line (u) with any line (v) except that v which it meets on $A = 0$; or a curve of the $(n-1)^{\text{th}}$ order, passing through $n(n-1)$ points (p), and meeting each of the $2n$ given lines (uv) $(n-1)$ times.

In like manner, if λ' be properly determined,

$$U_n' = v_1 v_2 \ldots v_e u_{e+1} u_{e+2} \ldots u_n - \lambda' u_1 u_2 \ldots u_e v_{e+1} v_{e+2} \ldots v_n = 0 = U'_{n-1} A$$

exhibits another such curve containing $n(n-1)$ points (p) and the whole of $A = 0$. This differs from $U_n = 0$ in that some functions (u), say the first e in U_n, have changed places in the equation with the corresponding (v).

By making all possible exchanges of r functions u for the corresponding r functions v in $U_n = 0$, for every value of $r \not> \frac{1}{2}n$, we shall thus obtain 2^{n-1} distinct curves (U_{n-1}) of the $(n-1)^{\text{th}}$ order, of which each will pass through a different set of $n(n-1)$ points (p).

3. Let any two such curves (U_{n-1}) be thus exhibited:

$$U_n = u_1 u_2 \ldots . u_n - \lambda v_1 v_2 \ldots . v_n = 0 = U_{n-1} A = M - \lambda N,$$

$$U_n' = v_1 \ldots v_e u_{e+1} \ldots u_n - \lambda' u_1 \ldots u_e v_{e+1} \ldots v_n = 0 = U'_{n-1} A = M' - \lambda' N'.$$

U_{n-1} and U'_{n-1} have in common $\{e(e-1) + (n-e)(n-e-1)\}$ points (p), and will in general intersect mutually in other $\{(n-1)^2 - e(e-1) - (n-e)(n-e-1)\}$ points not (p) nor (ϖ) At all these, whether (p) or not (p), the two equations

$$\lambda' M N' = \lambda M' N,$$

$$M' M = \lambda\lambda' N' N,$$

will be satisfied: and at the points not (p), the two following must hold good; for no factor has been removed from the above equations which can vanish at any of them, since *e.g.* $u_{e+1} = 0$ can meet neither U_{n-1} nor U'_{n-1} elsewhere than in $(n-1)$ points (p), and both U_n and U'_n contain every

point in $A = 0$:

$$U_{2e} = (u_1 u_2 \ldots u_e)^2 - \lambda : \lambda' (v_1 v_2 \ldots v_e)^2 = 0 = U_e U_{e-1} A,$$

$$U_{2(n-e)} = (u_{e+1} u_{e+2} \ldots u_n)^2 - \lambda\lambda' (v_{e+1} v_{e+2} \ldots v_n)^2 = 0 = U_{n-e} U_{n-e-1} A;$$

where $U_e = 0$ differs from $U_{e-1} A = 0$, as $U_{n-e} = 0$ from $U_{n-e-1} A = 0$ only in the sign of the constant multiplier in the second member.

All the points of intersection of the curves U_{2e} and $U_{2(n-e)}$ will satisfy the two equations before the last written, and consequently the pair

$$(M + \lambda N)(M - \lambda N) = 0,$$
$$(M' + \lambda' N')(M' - \lambda' N') = 0.$$

$U_e = 0$ and $U_{n-e} = 0$ must have, as well as $U_{e-1} A = 0$ and $U_{n-e-1} A = 0$, their multipliers $(\lambda : \lambda')^{\frac{1}{2}}$ and $(\lambda\lambda')^{\frac{1}{2}}$ preceded by like signs; otherwise it would follow that the intersections of the latter pair, and consequently all the line $A = 0$, are to be found upon the curves

$$M + \lambda N = 0 \text{ and } M' + \lambda' N' = 0,$$

which is impossible. Hence $U_e = 0$ and $U_{n-e-1} A = 0$, as also $U_{e-1} A = 0$ and $U_{n-e} = 0$, will have those multipliers preceded by unlike signs.

4. Thus all the points of intersection of $U_e = 0$ and $U_{n-e} = 0$, two given curves of the e^{th} and $(n-e)^{\text{th}}$ orders, will be upon both the curves

$$U_n = 0 = M - \lambda N = U_{n-1} A,$$
$$U_n' = 0 = M' - \lambda' N' = U'_{n-1} A,$$

and none of those points are either (p) or (ϖ); for U_e and U_{n-e} have none of these in common: therefore U_e and U_{n-e} contain each *not less* than $e(n-e)$ of the intersections, not (p), of U_{n-1} and U'_{n-1}.

U_e traverses $e(e-1)$ of the points (p) common to U_{n-1} and U'_{n-1}, as is evident from inspection in (3); and U_{n-e} contains other $(n-e)(n-e-1)$ of the same common points; wherefore U_e passes through *not more* than $\{e(n-1) - e(e-1) =\} \, e(n-e)$, and U_{n-e} through *not more* than

$$\{(n-1)(n-e) - (n-e)(n-e-1) =\} \; e(n-e)$$

of the intersections not (p) of U_{n-1} and U'_{n-1}. Therefore all the intersections, and no points besides, of U_e and U_{n-e} are points, not (p), common to U_{n-1} and U'_{n-1}.

It is also evident from inspection in (3) that U_{e-1} contains the above-mentioned $e(e-1)$ points (p) upon U_e, and that

U_{n-e-1} contains the aforesaid $(n-e)(n-e-1)$ points (p) upon U_{n-e}, that are among the intersections of U_{n-1} and U'_{n-1}: therefore U_{e-1} and U_{n-e-1} intersect in neither more nor less than $(e-1)(n-e-1)$ points not (p) that are common to U_{n-1} and U'_{n-1}. Thus every point not (p) in which these latter curves meet each other is either among the intersections of U_e with U_{n-e}, or among those of U_{e-1} with U_{n-e-1}; for U_e meets U_{e-1}, as U_{n-e} meets U_{n-e-1}, only in points (p), and

$$e(n-e)+(e-1)(n-e-1)=(n-1)^2-e(e-1)-(n-e).(n-e-1),$$

the exact number of the points not (p) in question.

5. We have proved that the four curves $[e > 0, < n]$

$$U_{n-1}=0,\quad U'_{n-1}=0,\quad U_e=0,\quad U_{e-1}=0,$$

have in common $e(e-1)$ points (p); that the four

$$U_{n-1}=0,\quad U'_{n-1}=0,\quad U_{n-e}=0,\quad U_{n-e-1}=0,$$

all meet in other $(n-e)(n-e-1)$ points (p); that the four

$$U_{n-1}=0,\quad U'_{n-1}=0,\quad U_e=0,\quad U_{n-e}=0,$$

traverse the same $e(n-e)$ points not (p); and that the four

$$U_{n-1}=0,\quad U'_{n-1}=0,\quad U_{e-1}=0,\quad U_{n-e-1}=0,$$

all intersect in other $(e-1)(n-e-1)$ points not (p); these being all given curves of the orders indicated by the sub-indices.

Inspection shews, in (3), that $U_e = 0$ contains, besides the points above enumerated, e of the n given points ϖ in $A=0$, and that the remaining $(n-e)$ are on U_{n-e}.

6. Since there are $2n(n-1)$ points (p) and 2^{n-1} curves (U_{n-1}), there will pass through every such point p' 2^{n-2} different curves (U_{n-1}); and for every pair of these passing through p' there go through the same point either a pair U_e and U_{e-1}, or a pair U_{n-e} and U_{n-e-1}; so that of the different curves above discussed there meet not less than 2^{2n-4} in p', in addition to the pair (uv) of lines that intersect there.

7. It is to be remarked that the curves in (3), U_{2e} and $U_{2(n-e)}$, or

$$U_{2e} = (u_1u_2\ldots u_e)^2 - \lambda:\lambda'(v_1v_2\ldots v_e)^2 = 0 = U_eU_{e-1}A,$$

$$U_{2(n-e)} = (u_{e+1}u_{e+2}\ldots u_n)^2 - \lambda\lambda'(v_{e+1}v_{e+2}\ldots v_n)^2 = 0 = U_{n-e}U_{n-e-1}A,$$

are curves of the form considered in (1),

$$p^2P - q^2Q = 0,$$

where $p = 0$ and $q = 0$ intersect in a point for whose coordinates $P : Q$ is positive and determinate. $U_{2e} = 0$ has e^2 such points, viz. $e(e-1)$ points (p) and e points (ϖ): $U_{2(n-e)} = 0$ has $(n-e)^2$ such points, viz. $(n-e)(n-e-1)$ points (p) and $(n-e)$ points (ϖ). At all these the two branches of the last-named curves have the property, that their tangents complete at each of them, whether (p) or (ϖ), an harmonic pencil with the pair $(u_e v_e)$ or $(u_e v_m)$ that intersect there. Consequently, in $U_{2e} = 0$, the branch $U_e = 0$ (as well as the branch $U_{n-e} = 0$ in $U_{2(n-e)} = 0$) touches, at every point ϖ_e in $A = 0$ which it contains, the line, $H = 0$, which completes a harmonic pencil with A and the pair $u_e v_e$; and, at every point p' found in it, touches a line which completes an harmonic pencil with the tangent of $U_{e-1} = 0$ (or $U_{n-e-1} = 0$) and the pair $u_t v_r$ which meet at p'.

8. When $e = 1$ (or $= n - 1$), U_e (or U_{n-e}) contains (5) no point (p), and is nothing else than the harmonical H, determined by A and the pair of lines whose equations appear transposed in $U'_{n-1}A = 0$ as compared with $U_{n-1}A = 0$, in (3).

In this case U_{e-1} (or U_{n-e-1}) represents a constant, and has no geometrical signification.

9. Suppose $n = 3$ and $e = 1$. U_{n-1} and U'_{n-1} (3) now represent any two of the four conic sections which pass (2) each through a different set of six points (p). U_{n-e} (3) is a given conic which meets those two in all their intersections, two (p) and two not (p); (5): it is met in both points (p) by the line U_{n-e-1}, and its tangent at either of them completes (7) with this line and the pair (uv) there intersecting an harmonic pencil: it is met in the two points not (p) by U_e, which is the harmonical (H) determined by A and the pair (uv) whose equations appear transposed in U'_{n-1} compared with U_{n-1}: and, finally, U_{n-e} contains two points (ϖ) which are not on U_e, and touches at them the harmonical (H) there determined by A and the other two pairs (uv).

10. It is evident, if λ be the constant already determined (3) to fulfil the condition

$$U_n = U_{n-1}A = 0 = M - \lambda N,$$

that the curve of the $2n^{\text{th}}$ order

$$U_{2n} = u_1^2 u_2^2 \dots u_n^2 - \lambda^2 v_1^2 v_2^2 \dots v_n^2 = 0 = V_n U_n = V_n U_{n-1} A = M^2 - \lambda^2 N^2,$$

consists of two branches $V_n = 0$ and $U_{n-1}A = 0$, which are *synharmonicals* (1) in respect of every pair $u_e v_e$ or $u_e v_m$ that

can be selected from *opposite* sides of $M^2 = \lambda^2 N^2$. $V_n = 0$ is therefore a given curve of the n^{th} order, which contains the n given points (ϖ), and touches the n harmonicals (H), synharmonic with A in respect of the n pairs $(u_e v_e)$; and which is synharmonic with $U_{n-1} = 0$ in respect of all the pairs $(u_e v_m)$ which determine its $n(n-1)$ intersections with U_{n-1}.

In like manner, if λ' be the constant already found, fulfilling the condition

$$U_n' = U'_{n-1}A = 0 = M' - \lambda' N',$$

$$U'_{2n} = v_1^2 \ldots v_e^2 u_{e+1}^2 \ldots u_n^2 - \lambda'^2 u_1^2 \ldots u_e^2 v_{e+1}^2 \ldots v_n^2 = 0$$

$$= V_n' U_n' = V_n' U'_{n-1} A = M'^2 - \lambda'^2 N'^2,$$

consists of two branches synharmonic in respect of every pair selected from opposite sides of $M'^2 = \lambda'^2 N'^2$. And thus we see given 2^{n-1} curves (V_n) of the n^{th} order that are synharmonic, ($V_n^{(m)}$ with $U_{n-1}^{(m)}$), with the 2^{n-1} given curves (U_{n-1}) in respect of $n(n-1)$ pairs of lines (uv) which determine the intersections of $V_n^{(m)}$ and U_{n-1}^{m}, and which all touch the same n harmonicals (H), at the same n points (ϖ).

11. If we select any pair of these curves (V_n) in which any e pairs $(u_a v_a)$ of lines appear transposed, on comparison of the two equations, as *e.g.* the first e pairs that are exhibited in $V_n = 0$, we have

$$V_n = u_1 u_2 \ldots u_n + \lambda v_1 v_2 \ldots v_n = 0 = M + \lambda N,$$

$$V_n' = v_1 \ldots v_e u_{e+1} \ldots u_n + \lambda' u_1 \ldots u_e v_{e+1} \ldots v_n = 0 = M' + \lambda' N'.$$

V_n and V_n' have in common $\{e(e-1) + (n-e)(n-e-1)\}$ points (p), and $2n$ points (ϖ), since they have a common tangent (H) at n points (ϖ); they will therefore intersect in general in other $\{n^2 - 2n - e(e-1) - (n-e)(n-e-1)\}$ points not (p) nor (ϖ).

At all these, whether (p) or (ϖ) or otherwise, the two equations

$$\lambda' M N' = \lambda M' N,$$

$$M M' = \lambda\lambda' N N',$$

will be satisfied; and at the points not (p) of the intersection of V_n and V_n', the following must hold good; for no factor has been removed that can vanish at any of them, since *e.g.* $u_{e+1} = 0$ can meet neither curve, but at one point (ϖ), and $(n-1)$ points (p):

$$U_{2e} = (u_1 u_2 \ldots u_e)^2 - \lambda : \lambda' (v_1 v_2 \ldots v_e)^2 = 0 = U_e U_{e-1} A,$$

$$U_{2(n-e)} = (u_{e+1} u_{e+2} \ldots u_n)^2 - \lambda\lambda' (v_{e+1} v_{e+2} \ldots v_n)^2 = 0 = U_{n-e} U_{n-e-1} A;$$

loci before obtained in (3), and known to contain $A = 0$.

The points common to U_e and U_{n-e-1}, or those common to U_{n-e} and U_{e-1}, are the points that must satisfy, as has been shewn, both the equations $V_n = 0$ and $V_n' = 0$. Of these the former pair meet V_n and V_n' in *at least* $e(n-e-1)$ points not (p); for the pair have no point, (p) or (ϖ) in common. U_e contains $e(e-1)$ points (p) of the intersections of V_n and V_n', as is evident from inspection, and (7) touches at e points (ϖ) their common tangents (H) at those points: it goes, therefore, through *not more* than

$$\{ne - e(e-1) - 2e =\}\ e(n-e-1)$$

of their mutual points not (p).

U_{n-e-1} traverses $(n-e)(n-e-1)$ of these intersections (p), and contains therefore *not more* than

$$\{n(n-e-1) - (n-e)(n-e-1) =\}\ e(n-e-1)$$

of those not (p).

In the same way it can be shewn that U_{n-e} and U_{e-1} meet in neither less nor more than $(n-e)(e-1)$ points not (p) of the intersections of V_n and V_n'; the former meeting both in $n-e$ points of contact with as many tangents (H) in the line A, and containing $(n-e)(n-e-1)$, as U_{e-1} contains $e(e-1)$, of their intersections (p). Hence, since

$$e(n-e-1) + (n-e)(e-1) = n^2 - 2n - e(e-1) - (n-e)(n-e-1),$$

it appears that all the points not (p) or (ϖ) that are common to V_n and V_n', are either among the intersections of U_e with U_{n-e-1}, or among those of U_{n-e} with U_{e-1}.

12. We have now proved that the six curves $(e > 0, < n)$,

$$V_n = 0,\ V_n' = 0,\ U_e = 0,\ U_{e-1} = 0,\ U_{n-1} = 0,\ U'_{n-1} = 0,$$

have in common $e(e-1)$ points (p); that the six

$$V_n = 0,\ V_n' = 0,\ U_{n-e} = 0,\ U_{n-e-1} = 0,\ U_{n-1} = 0,\ U'_{n-1} = 0,$$

contain alike other $(n-e)(n-e-1)$ points (p); that the four

$$V_n = 0,\ V_n' = 0,\ U_e = 0,\ U_{n-e-1} = 0,$$

intersect mutually in $e(n-e-1)$ points not (p); that the four

$$V_n = 0,\ V_n' = 0,\ U_{n-e} = 0,\ U_{e-1} = 0,$$

meet in other $(n-e)(e-1)$ points not (p); that the three

$$V_n = 0,\ V_n' = 0,\ U_e = 0,$$

have e common tangents (H) at the same e points (ϖ); and that

$$V_n = 0,\ V_n' = 0,\ U_{n-e} = 0,$$

have other $(n-e)$ common tangents (H) at the remaining points (ϖ); where V_n differs from $U_{n-1}A$, as V_n' from $U_{n-1}A$, only in the sign of the constant in the second member of the equation

$$M = \pm \lambda N,$$

and V_n and V_n' are *any pair* of the 2^{n-1} curves (V_n), in whose equations when compared there appears a transposition of e of the linear functions $(u_a v_a)$.

13. Suppose $n = 3$, $e = 1$. $V_n = 0$, and $V_n' = 0$ are now given curves of the third order, which touch alike the three harmonicals (H) at the three given points (ϖ), and meet each its synharmonic (U_{n-1}, U'_{n-1}) at the six angles of the hexagon inscribed therein by the six given lines (uv), and which intersect at the pair of angles which are common to these conics. Both V_n and V'_n have double contact at the same two points (ϖ) with the conic U_{n-e}, which passes through the four intersections of U_{n-1}, and U'_{n-1}; and intersect in their only remaining, or ninth common point, upon the line U_{n-e-1}, which meets at both of two points (p) all the four V_n, V'_n, U_{n-1}, U'_{n-1}. At the third point ϖ, not upon the conic u_{n-e-1}, the harmonical U_e is a tangent to both V_n and V'_n: this line meets the two curves of the third order again at their common intersection with the chord U_{n-e-1}, and afterwards passes through the two intersections not (p) of the conics U_{n-1} and U'_{n-1}.

14. The constancy of the multiplier, in the second member of the equations of the different curves we have examined, implies that the product of perpendiculars from any point in any of them, U_n, on the first, third, fifth, &c. of its consecutive chords (uv) through the points (p) on it, has a constant ratio to the product of perpendiculars from the same point on the second, fourth, &c. of those chords. In two synharmonic curves, or V_n and V_{n-1}, or U_e and U_{e-1}, these ratios differ always in sign. If then a transversal meet the curve U_{n-1}, (3) in any $(n-1)$ points of which P and Q are a pair, and its synharmonic curve V_n in any pair out of n points, P′ and Q′, and their $2n$ common chords $u_1u_2\ldots u_n$, $v_1v_2\ldots v_n$, in the points $\mathrm{u}_1\mathrm{u}_2\ldots\mathrm{u}_n$, $\mathrm{v}_1\mathrm{v}_2\ldots\mathrm{v}_n$, we shall have

$$\mathrm{Pu_1P_2Pu_3.\ldots.Pu_nQv_1Qv_2Qv_3.\ldots.Qv_n}$$
$$= \mathrm{Pv_1Pv_2Pv_3.\ldots.Pv_nQu_1Qu_2Qu_3.\ldots.Qu_n},$$
$$\mathrm{P'u_1P'u_2P'u_3.\ldots.P'u_nQ'v_1Q'v_2Q'v_3.\ldots.Q'v_n}$$
$$= \mathrm{P'v_1P'v_2P'v_3.\ldots.P'v_nQ'u_1Q'u_2Q'u_3.\ldots.Q'u_n}.$$

If $p_1p_2 \dots p_n$ be perpendiculars from any point P in U_{n-1} upon $u_1u_2 \dots u_n$, and $q_1q_2 \dots q_n$ be those from the same point upon $v_1v_2 \dots v_n$; and if from any point P' in V_n be let fall the perpendiculars $p_1'p_2' \dots p_n'$, $q_1'q_2' \dots q_n'$, upon the same $2n$ lines, we shall have

$$p_1p_2 \dots p_n \, q_1'q_2' \dots q_n' + p_1'p_2' \dots p_n' \, q_1q_2 \dots q_n = 0.$$

15. We have to add the curves (V_n) to those already enumerated in Art. (6). Since one of these passes through every point (p) for one of the curves (U_{n-1}), there will pass through any one point, p', as many of (V_n) as of (U_{n-1}), *i.e.* 2^{n-2}, which raises the number of the curves enumerated as meeting in p' to $2^{2n-4} + 2^{2n-2} = 2^{n-2}.(2^{n-2} + 1)$.

16. The consideration of every case of $n = 3$, and $e = 1$, will bring before us six different pairs U_{n-1}, and U'_{n-1}, and it will be six times true that the line $U_e(5)$, one of the harmonicals (H), twice meets three conics in a point, U_{n-1}, U'_{n-1}, and U_{n-e}; by which, as there are only three of these harmonicals, we see that in each of the lines (H) are four points, in every one of which it meets three given conics.

It will also be six times true, that a line U_e (12), or one of the harmonicals (H), meets two of the curves (V_n) at one of their intersections; by which we see that there are two points in each of the three lines H, at which two distinct pairs of the curves (V_3) of the third order intersect. It is (12) also six times true, that a distinct conic (U_{n-e}) is twice in contact at two points (ϖ) with lines (H), *i.e.* there are twelve of these contacts; so that four must be at each point ϖ; or each line (H) touches in the same point (ϖ) the four curves (V) of the third order, and four of the six conics (U_{n-e}).

Further, it is six times true, that (12) five curves, namely, two of the third order (V_n), and the three conics U_{n-1}, U'_{n-1}, and U_{n-e}, meet a certain line U_{n-e-1} in two of the twelve points (p), which verifies the enumeration in (15); whence we see that through each of the twelve points of intersection of the three given pairs (uv) that are not on A, go three given conics and two given curves of the third order.

17. The theorems which follow are now established by the development of a simple conception, which has not, so far as I know, been turned to account by any other writer; and I may perhaps venture to say, that the properties have never in their present generality been delivered before.

THEOREM I. If any $2n$ right lines (u) meet by pairs in n points (ϖ) of a given line A, and intersect in separate pairs in certain points (p), not in A, there are given 2^{n-1} curves (U_{n-1}) of the $(n-1)^{\text{th}}$ order, in each of which the $2n$ right lines form a net-work of $n.(n-1)$ intersections (p) upon the curve, so that 2^{n-2} of these curves (U_{n-1}) pass through every point (p).

THEOREM II. With every one of the curves (U_{n-1}) is given a curve (V_n) of the n^{th} order, which meets that one in $n.(n-1)$ points (p), and is synharmonic with it in respect of the n pairs (u) whose intersections determine those $n.(n-1)$ points (p). All these 2^{n-1} curves (V_n) are synharmonic with A in respect of the n pairs (u), so that at each point (ϖ) are 2^{n-1} curves (V_n) in contact, while 2^{n-2} of the same curves pass through every point (p).

THEOREM III. Any two of the curves (U_{n-1}) have in common

$$\{e.(e-1)+(n-e).(n-e-1)\}$$

of the points (p) $[e>0, <n]$; and their remaining intersections, not (p), lie all upon both of two given loci; one $(U_e U_{e-1})$, consisting of a curve of the e^{th} and a curve of the $(e-1)^{\text{th}}$ order, which contain each $e.(e-1)$ of the abovenamed common points (p), and are synharmonicals with respect to the pairs of lines (u) that meet in those $e.(e-1)$ points; the other $(U_{n-e} U_{n-e-1})$ two curves of the $(n-e)^{\text{th}}$ and $(n-e-1)^{\text{th}}$ orders, both containing the remainder of those common points (p) and synharmonicals with respect to the lines (u) that intersect thereat: U_e and U_{n-e} intersect in $e.(n-e)$ of the points, not (p), common to the pair of curves (U_{n-1}), while the remainder of those points, not (p), are the intersections of U_{e-1} and U_{n-e-1}, the other two curves of the loci.

THEOREM IV. Any two of the curves (V_n) will have in common $\{e.(e-1)+(n-e).(n-e-1)\}$ points (p) $[e>0, <n]$, and $2n$ common points, considering each contact as a double point, on their n common tangents (H). Their remaining intersections, not (p) or (ϖ), will lie all upon both of two given loci: one $(U_e U_{e-1})$, the other $(U_{n-e} U_{n-e-1})$, the first pair synharmonicals at $e.(e-1)$, the second synharmonicals at $(n-e).(n-e-1)$, points (p) common to the pair (V_n), in respect of the lines (u) that determine those points (p); while the points, not (p) or (ϖ), common to the pair (V_n) are $e.(n-e-1)$ intersections of U_e with U_{n-e-1}, and $(n-e).(e-1)$ intersections of U_{n-e} with U_{e-1}. The pair (V_n) have common contact with U_e at e points ϖ; and common contact with U_{n-e} at the remaining points (ϖ); the common tangent of three curves at each point (ϖ) being the line (H) that completes an harmonic pencil there with A and a pair of the lines (u).

THEOREM V. All the curves above described, of the n^{th}, $(n-1)^{\text{th}}$, $(n-e)^{\text{th}}$, $(n-e-1)^{\text{th}}$, e^{th} and $(e-1)^{\text{th}}$ orders, have the property, that if perpendiculars be let fall from any point in any of them upon all the chords (u) meeting it in points (p), the product of those let fall on that half of them, no two of which meet upon the curve, will be to that of those let fall upon the other half, everywhere in the same ratio for the same curve; and in the pairs of synharmonic curves, those ratios will differ always in sign.

18. All the above theorems admit of transformation by the doctrine of reciprocal polars; or if we prefer to employ

the elegant method of line-coordinates of Professor Plücker, we can obtain the corresponding reciprocal polar theorems by the exact path above pursued.

19. The reader will not fail to recognise, in what has preceded, the converted theorems of Pascal and Brianchon. The slightest change in this mode of investigation brings out these propositions instantly in their usual form.

Let $2(n+1)$ chords $u_0u_1\ldots u_n$, $v_0v_1\ldots v_n$, so traverse in any way a curve of the $(n)^{\text{th}}$ order, U_n, that u_e shall meet thereon all of $v_0v_1\ldots$ except v_e, and so for the rest, inscribing a network of $n.(n+1)$ angles; then

$$U_{n+1} = u_0u_1\ldots u_n - \lambda v_0v_1\ldots v_n = 0 = U_nA,$$

if λ be properly determined; where $A = 0$ is a given right line, and $U_n = 0$ is a given curve of the n^{th} order, containing the $(n+1)$ intersections of u_e with v_e, &c. The constant λ is to be so determined that $U_{n+1} = 0$ shall contain some $\{n.(n+1)+1\}^{\text{th}}$ point of $U_n = 0$.

We have thus arrived at the following:

THEOREM VI. If $n.(n+1)$ points in a curve of the n^{th} order be traversed in any way by $(2n+2)$ chords, so that each point is traversed by some two of them, a straight line is given in which each chord meets one that it does not meet upon the curve.

If our variables had been the coordinates of a line, instead of those of a point, the same steps would have conducted to the corresponding reciprocal theorem.

20. Another (!) demonstration of the theorems of Pascal and Brianchon (how many does this make?) may be obtained as follows, with equal rapidity and ease, by the proof of a more general property.

THEOREM VII. If $2n$ points in a conic section be numbered in any random order $12\ldots(2n)$, and traversed in that order by the $2n$ consecutive chords $u_{12} = 0$, $u_{23} = 0$, $u_{34} = 0$, &c.; there is given a curve of the $(n-2)^{\text{th}}$ order, in which lie the intersections of every e^{th} chord with the $(e+3)^{\text{th}}$, the $(e+5)^{\text{th}}$, the $(e+7)^{\text{th}}$, &c., $(n-2)$ intersections of such e^{th} chord.

This is proved by inspection of the equation

$$U_n = u_{12}u_{34}\ldots u_{2n-1,\,2n} - \lambda u_{23}u_{45}\ldots u_{2n,\,1} = 0 = U_{n-2}C,$$

if λ be supposed so determined that $U_n = 0$ shall pass through π, some $(2n+1)^{\text{th}}$ point of the given conic $C = 0$.

If we suppose λ so determined that U_n shall pass through some $\{n.(n-2)+1\}^{\text{th}}$ point of $U_{n-2} = 0$, inspection of the same equation will demonstrate the following:

THEOREM VIII. If $n.(n-2)$ points on a curve of the $(n-2)^{\text{th}}$ order are traversed in any way by $2n$ chords so that each point is traversed by some two of them, these chords are the sides of a $2n$-gon inscribed in a conic section, whose angles are not upon the curve.

The last two theorems can be easily transformed by the theory of reciprocal polars. Some time after I obtained them, I found that both these, and Theorem VI., as well as the theorem proved in the first part of Art. (2), are in fact cases (though I am not aware that they have ever been explicitly deduced from it) of the following more general and most elegant theorem of Professor Plücker, ('Theorie der algebraischen Curven,' p. 12; *Bonn*, 1839).

"If $\{np - \frac{1}{2}(p-1).(p-2)$ of the n^2 intersections of two curves of the n^{th} order lie upon a curve of the p^{th} order, a curve of the $(n-p)^{\text{th}}$ order will pass through $n.(n-p)$ of the remaining intersections."

The constancy of the multiplier λ implies the property expressed in

THEOREM IX. If from any point of the conic section, or of the corresponding curve of the $(n-2)^{\text{th}}$ order, which is defined in the two preceding theorems, perpendiculars be let fall on the $2n$ chords of the same theorems, the product of those let fall on that half of the chords no two of which meet upon the curve, will be to that of those let fall on the other half everywhere in the same ratio.

21. If in Theorem VII. there be given on the conic section only $(2n-b)$ points, say $b = 3$, and if b tangents be drawn to the conic at any b of the $(2n-b)$ points, say at those numbered 2, 3, 5; the curve of the n^{th} order,

$$\begin{aligned} U_n' = {} & u_{12}u_{23}u_{34}(u_{45} + \theta u_{56})\,u_{67}u_{89}\ldots u_{2n-4,\,2n-3} \\ & - \lambda(u_{12} + \iota u_{23})(u_{23} + \kappa u_{34})\,u_{45}u_{56}u_{78}\ldots u_{2n-3,\,1} = 0, \end{aligned}$$

passes through all the $2n - b$ points, and touches the b tangents $(u_{12} + \iota u_{23}) = 0$, $(u_{23} + \kappa u_{34}) = 0$, $(u_{45} + \theta u_{56}) = 0$, at the points 2, 3, and 5, whatever be $\theta\iota\kappa\lambda$. These $(b+1)$ arbitraries can be so determined, that U_n' shall still pass through $(2n+1)$ points of the conic, in which case we shall have again

$$U_n' = U'_{n-2}C = 0;$$

so that U'_{n-2} is a given curve of the $(n-2)^{\text{th}}$ order, on which every one of the n lines in the first member of $U_n' = 0$ meets $(n-2)$ of the lines in its second member. This is only saying, that in the above equation $U_n = U_{n-2}C = 0$, any two of the $2n$ points in the conic may coalesce, as the points 2 and 3, and the equation will still be true, if $u_{23} = 0$ be

understood to represent an evanescent chord, or the tangent to the conic at the intersection of $u_{12} = 0$ and $u_{34} = 0$. We may thus consider any inscribed polygon of $2n - b$ sides, along with *any* b tangents at its angles, to form one of the $2n$-laterals in the conic of which we are now discussing some of the properties.

22. We may in like manner obtain, for every distinct way in which $2n$ consecutive chords can traverse $2n$ given points in a conic section, a distinct curve, $U_{n-2} = 0$, on which each of those chords meets $n - 2$ others of the $2n$ which it does not meet upon the conic. This gives $3.4.5...(2n - 1)$, as the number of different curves (U_{n-2}) of the $(n - 2)^{\text{th}}$ order. Of $n.(2n - 1)$ chords that can be drawn connecting the $2n$ points, each one meets $2.(2n - 2)$ others upon the conic, the number of the points (p) of their mutual intersections, not upon the conic, being

$$\tfrac{1}{2}\{n.(2n-1)-2.(2n-2)-1\}(2n^2-n)=\tfrac{1}{2}\{(n-2).(2n-1)+1\}(2n^2-n).$$

If h be the number of these curves (U_{n-2}) which pass through every point (p),

$$\tfrac{1}{2}.h.\{(n-2).(2n-1)+1\}(2n^2-n) = 3.4.5...(2n-1).n.(n-2),$$

since there are $n.(n - 2)$ points (p) upon every curve (U_{n-2});

$$\text{therefore } h = \{1.2.3.4...(2n-2)\}.\frac{n-2}{(n-2).(2n-1)+1}:$$

and this is exactly the number of curves of the $(n - 2)^{\text{th}}$ order passing through every point (p), and containing each $n.(n-2)$ intersections of the sides of a given closed $2n$-gon in the conic C. When $n = 3$, $h = 4$, which is the exact number of the lines (U_{n-2}) that pass through each of the 45 points (p); nor can more than 4 such lines be found. But when $n > 3$, the number h is far from comprising all the curves of the $(n - 2)^{\text{th}}$ order which pass through every point (p), each containing $n.(n - 2)$ points (p).

We have supposed that the $2n$ points of the conic section are connected by $2n$ *consecutive* chords, forming one closed polygon of $2n$ sides. But these points may be so connected as to form several polygons of an even number of sides. For instance, when $n = 6$, any permutation $1326540798\alpha\beta$ exhibited in the first member of $U_n = 0$, may give rise to a second member in the following distinct ways; putting $0, \alpha, \beta$, for $10, 11, 12$:

$$
\begin{aligned}
U_n &= u_{13}u_{26}u_{54}u_{07}u_{98}u_{\alpha\beta} - \lambda\, u_{32}u_{65}u_{40}u_{79}u_{8\alpha}u_{\beta_1} = 0 = U_{n-2}\ \ C,\\
U_n' &= u_{13}u_{26}u_{54}u_{07}u_{98}u_{\alpha\beta} - \lambda_1 u_{32}u_{65}u_{41}u_{79}u_{8\alpha}u_{\beta_0} = 0 = U_{n-2}'\ C,\\
U_n'' &= u_{13}u_{26}u_{54}u_{07}u_{98}u_{\alpha\beta} - \lambda_2 u_{32}u_{61}u_{40}u_{79}u_{8\alpha}u_{\beta_5} = 0 = U_{n-2}''\ C,\\
U_n''' &= u_{13}u_{26}u_{54}u_{07}u_{98}u_{\alpha\beta} - \lambda_3 u_{32}u_{65}u_{40}u_{71}u_{8\alpha}u_{\beta_9} = 0 = U_{n-2}'''\ C,\\
U_n'''' &= u_{13}u_{26}u_{54}u_{07}n_{98}u_{\alpha\beta} - \lambda_4 u_{32}u_{61}u_{40}u_{75}u_{8\alpha}u_{\beta_9} = 0 = U_{n-2}''''\ C.
\end{aligned}
$$

Each of these curves of the 6^{th} order passes through 12 points of the conic, and by properly determining the arbitrary constant λ, we obtain five distinct curves (U_4, &c.) of the fourth order, each passing through a different set of $\{n.(n-2)=\}$ 24 points (p). The first curve, U_{n-2} of the fourth order, is one of those already enumerated, and contains 24 of the mutual intersections of a closed polygon of 12 sides inscribed in the conic C; the next contains the same number of intersections of two closed hexagons whose angles are the same 12 points; the third and fourth pass each through those of the sides of a different quadrilateral and octagon; the fifth through those of three quadrilaterals. None of the four last curves of the fourth order is identical with any of the curves ($U_4 = U_{n-2}$) already considered.

[*To be continued.*]

ON CERTAIN PROPERTIES OF SURFACES OF THE SECOND ORDER.

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1. In the last number of the *Math. Journal* (vol. v. p. 69) I established an extension of MacCullagh's modular generation of surfaces of the second order (*Proceedings of the Royal Irish Academy*, vol. II. p. 471). That extension consisted in connecting by a right line any point upon the generated surface with *two* points, one on *each* focal curve; or, when regarded from a more general point of view, in connecting, by a common tangent chord, any point upon the generated surface with two points, one on *each* of two confocal surfaces. We have seen that it brought into evidence the many beautiful known properties of surfaces of the second degree, while at the same time it indicated between them a natural order of dependence. In that investigation, however, the properties likely to arise from the consideration of the two confocal surfaces, touched by the common tangent chord, were not

discussed: the present article seeks to supply the deficiency, and we shall see that it leads to many very interesting and unexpected results. The most immediate connection which our method suggests, between the original and any two confocal surfaces, is obviously through the enveloping cones of the latter; and this is consequently the direction in which I have sought to continue my former investigations. The intersection of any two confocal surfaces with the original determines, as is well known, a point upon its surface, and at this point I conceive the three normals drawn. In the point so determined, the common tangent chord is supposed to terminate, and to the three normals, as axes of coordinates, the equation of either confocal surface touched by the chord is supposed to be referred. This equation having been determined, the present investigations follow in detail. The subject, however, is not as yet by any means exhausted; since many directions obviously remain still unexplored, which appear likely to yield interesting and important results to adequate research.

Let (S) be any central surface of the second order, the equation of which referred to ordinary rectangular coordinates is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - b^2} + \frac{z^2}{a^2 - c^2} = 1.$$

Let (ρ, μ, ν) be the semi-major axes of the three confocal surfaces which intersect in the point (x', y', z'); then if the three normals at this point to the three confocal surfaces be selected for axes of coordinates, the equation of the surface (S) will become

$$(1)\quad \left(\frac{\xi^2}{\rho^2 - a^2} + \frac{\eta^2}{\mu^2 - a^2} + \frac{\zeta^2}{\nu^2 - a^2}\right)\left(\frac{\xi_0^2}{\rho^2 - a^2} + \frac{\eta_0^2}{\mu^2 - a^2} + \frac{\zeta_0^2}{\nu^2 - a^2} - 1\right)$$
$$= \left(\frac{\xi\xi_0}{\rho^2 - a^2} + \frac{\eta\eta_0}{\mu^2 - a^2} + \frac{\zeta\zeta_0}{\nu^2 - a^2} - 1\right)^2,$$

where (ξ_0, η_0, ζ_0) are the coordinates of the centre of the surface (S) referred to the new rectangular axes of coordinates. Now the equation

$$\frac{\xi^2}{\rho^2 - a^2} + \frac{\eta^2}{\mu^2 - a^2} + \frac{\zeta^2}{\nu^2 - a^2} = 0$$

is, as we have already seen (*Journal*, vol. v. p. 84), the equation of the cone which has its vertex at the point (x', y', z'), and circumscribes the surface (S) while the equation of the

plane of contact is

$$\frac{\xi\xi_0}{\rho^2 - a^2} + \frac{\eta\eta_0}{\mu^2 - a^2} + \frac{\zeta\zeta_0}{\nu^2 - a^2} = 1:$$

but in ordinary rectangular coordinates the equation of the plane of contact is

$$\frac{xx'}{a^2} + \frac{yy'}{a^2 - b^2} + \frac{zz'}{a^2 - c^2} = 1:$$

when referred to parallel axes of coordinates at the point (x', y', z') it becomes, since $x = x' - X$, $y = y' - Y$, $z = z' - Z$,

$$\text{(a)}\quad \frac{x'X}{a^2} + \frac{y'Y}{a^2 - b^2} + \frac{z'Z}{a^2 - c^2} = \frac{x'^2}{a^2} + \frac{y'^2}{a^2 - b^2} + \frac{z'^2}{a^2 - c^2} - 1;$$

consequently, since

$$\begin{aligned} X &= \cos\alpha \,.\, \xi + \cos\beta \,.\, \eta + \cos\gamma \,.\, \zeta, \\ Y &= \cos\alpha' .\, \xi + \cos\beta' .\, \eta + \cos\gamma' .\, \zeta, \\ Z &= \cos\alpha'' .\xi + \cos\beta'' .\eta + \cos\gamma'' .\zeta, \end{aligned}$$

where (α, β, γ) are the angles made by the axis of X with the axes of (ξ, η, ζ), &c., we can easily see that

$$(2)\quad \left\{ \begin{aligned} \frac{\xi_0}{\rho^2 - a^2} &= \frac{1}{A^2} \cdot \left\{ \frac{\cos\alpha . x'}{a^2} + \frac{\cos\alpha' y'}{a^2 - b^2} + \frac{\cos\alpha'' z'}{a^2 - c^2} \right\}, \\ \frac{\eta_0}{\mu^2 - a^2} &= \frac{1}{A^2} \cdot \left\{ \frac{\cos\beta . x'}{a^2} + \frac{\cos\beta' y'}{a^2 - b^2} + \frac{\cos\beta'' z'}{a^2 - c^2} \right\}, \\ \frac{\zeta_0}{\nu^2 - a^2} &= \frac{1}{A^2} \cdot \left\{ \frac{\cos\gamma . x'}{a^2} + \frac{\cos\gamma' y'}{a^2 - b^2} + \frac{\cos\gamma'' z'}{a^2 - c^2} \right\}, \end{aligned} \right.$$

where $$A^2 = \frac{x'^2}{a^2} + \frac{y'^2}{a^2 - b^2} + \frac{z'^2}{a^2 - c^2} - 1.$$

Hence, multiplying the first by ξ_0, the second by η_0, and the third by ζ_0, we shall have

$$(3)\quad \frac{\xi_0^2}{\rho^2 - a^2} + \frac{\eta_0^2}{\mu^2 - a^2} + \frac{\zeta_0^2}{\nu^2 - a^2} - 1 = \frac{1}{\dfrac{x'^2}{a} + \dfrac{y'^2}{a^2 - b^2} + \dfrac{z'^2}{a^2 - c^2} - 1},$$

therefore the equation (1) becomes

$$(4)\quad \frac{\xi^2}{\rho^2 - a^2} + \frac{\eta^2}{\mu^2 - a^2} + \frac{\zeta^2}{\nu^2 - a^2}$$

$$= \left(\frac{x'^2}{a^2} + \frac{y'^2}{a^2 - b^2} + \frac{z'^2}{a^2 - c^2} - 1 \right) \left(\frac{\xi\xi_0}{\rho^2 - a^2} + \frac{\eta\eta_0}{\mu^2 - a^2} + \frac{\zeta\zeta_0}{\nu^2 - a^2} - 1 \right)^2,$$

which is the form of the equation first given by MacCullagh, who also first observed that the equation

$$\frac{\xi^2}{\rho^2 - a^2} + \frac{\eta^2}{\mu^2 - a^2} + \frac{\zeta^2}{\nu^2 - a^2} = 0$$

indicates the circumscribing cone of the surface (S).

From the equations (2) it is easy to perceive that, if upon the axes (ρ, μ, ν) of the circumscribing cone the plane of contact intercept right lines respectively equal to $\rho_{,}$, $\mu_{,}$, and $\nu_{,}$, we shall then have

$$\frac{1}{\rho_{,}} = \frac{\xi_0}{\rho^2 - a^2}, \quad \frac{1}{\mu_{,}} = \frac{\eta_0}{\mu^2 - a^2}, \quad \frac{1}{\nu_{,}} = \frac{\zeta_0}{\nu^2 - a^2}.$$

We have not as yet, however, in either (1) or (4), found the ultimate form of the equation; for, as is well known, since

$$x'^2 = \frac{\rho^2\mu^2\nu^2}{b^2c^2}, \quad y'^2 = \frac{(\rho^2 - b^2)(\mu^2 - b^2)(\nu^2 - b^2)}{b^2(b^2 - c^2)},$$

$$z'^2 = \frac{(\rho^2 - c^2)(\mu^2 - c^2)(\nu^2 - c^2)}{c^2(c^2 - b^2)},$$

we shall have

$$(5) \quad \frac{x'^2}{a^2} + \frac{y'^2}{a^2 - b^2} + \frac{z'^2}{a^2 - c^2} - 1 = \frac{(\rho^2 - a^2)(\mu^2 - a^2)(\nu^2 - a^2)}{a^2(a^2 - b^2)(a^2 - c^2)}.*$$

Therefore the ultimate and most convenient form of the equation of the surface (S) is

$$(6) \quad \frac{\xi^2}{\rho^2 - a^2} + \frac{\eta^2}{\mu^2 - a^2} + \frac{\zeta^2}{\nu^2 - a^2}$$

$$= \frac{(\rho^2 - a^2)(\mu^2 - a^2)(\nu^2 - a^2)}{a^2(a^2 - b^2)(a^2 - c^2)} \left\{ \frac{\xi\xi_0}{\rho^2 - a^2} + \frac{\eta\eta_0}{\mu^2 - a^2} + \frac{\zeta\zeta_0}{\nu^2 - a^2} - 1 \right\}^2.$$

The equation of the plane of contact (a) also becomes

$$(\text{a}') \quad \frac{x'X}{a^2} + \frac{y'Y}{a^2 - b^2} + \frac{z'Z}{a^2 - c^2} = \frac{(\rho^2 - a^2)(\mu^2 - a^2)(\nu^2 - a^2)}{a^2(a^2 - b^2)(a^2 - c^2)}.$$

If (δ_1) denote the perpendicular upon this plane from the centre of the surface (S), we shall have

$$(\text{a}'') \quad \frac{1}{\delta_1^2} = \frac{x'^2}{a^4} + \frac{y'^2}{(a^2 - b^2)^2} + \frac{z'^2}{(a^2 - c^2)^2}.$$

This is evidently the generalized expression for the perpendicular from the centre upon the tangent plane to the surface:

* I have found that the equations (3) and (5) are the analytical solution of a theorem of great beauty and importance: their consideration I hope to resume on some future occasion.

now let (δ) denote the perpendicular from the point (x', y', z') upon the plane of contact (a′), and we shall have

$$(\text{a}''')\quad \frac{1}{\delta^2} = \frac{\xi_0^2}{(\rho^2 - a^2)^2} + \frac{\eta_0^2}{(\mu^2 - a^2)^2} + \frac{\zeta_0^2}{(\nu^2 - a^2)^2}:$$

but since from the equations (2) we have

$$A^4 . \left\{\frac{\xi_0^2}{(\rho^2 - a^2)^2} + \frac{\eta_0^2}{(\mu^2 - a^2)^2} + \frac{\zeta_0^2}{(\nu^2 - a^2)^2}\right\} = \frac{x'^2}{a^4} + \frac{y'^2}{(a^2 - b^2)^2} + \frac{z'^2}{(a^2 - c^2)^2},$$

we consequently have

$$(7)\quad \delta = \frac{(\rho^2 - a^2)(\mu^2 - a^2)(\nu^2 - a^2)}{a^2(a^2 - b^2)(a^2 - c^2)}\,\delta_1.$$

If the equation of a plane be $\varpi x + \varpi' y + \varpi'' z = 1$, and we substitute for (x, y, z) the coordinates of any point $(x'y'z')$ in space, we shall have

$$P = \frac{\varpi x' + \varpi' y' + \varpi'' z' - 1}{\sqrt{(\varpi^2 + \varpi'^2 + \varpi''^2)}},$$

where (P) indicates the perpendicular upon this plane from the point $(x'y'z')$. The equations (5) and (a″) evidently enable us to interpret the same general expression for equations of the second order; for the equation of the surface (S) being

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - b^2} + \frac{z^2}{a^2 - c^2} = 1,$$

we shall have

$$(8)\quad \frac{\dfrac{x'^2}{a^2} + \dfrac{y'^2}{a^2 - b^2} + \dfrac{z'^2}{a^2 - c^2} - 1}{\sqrt{\left(\dfrac{x'^2}{a^4} + \dfrac{y'^2}{(a^2 - b^2)^2} + \dfrac{z'^2}{(a^2 - c^2)^2}\right)}} = P,$$

(P) in the present instance indicating the perpendicular from any assumed point $(x'y'z')$ in space upon the plane of contact of the cone, which has the point $(x'y'z')$ for vertex and circumscribes the surface (S). This expression is perfectly general, and when suitably modified of course includes the case where (S) is a non-central surface: however, as these limiting cases contain many points of interest, we propose to consider them in detail in another portion of the present article. Let any right line (l) drawn from the point $(x'y'z')$ and intersecting the surface (S) make with the axes (ξ, η, ζ) the angles (ι, ι', ι'') and with the axes (x, y, z) the angles (ψ, ω, ϕ), from the equations (2) we can easily deduce

$$(9)\quad \frac{\xi_0 \cos\iota}{\rho^2 - a^2} + \frac{\eta_0 \cos\iota'}{\mu^2 - a^2} + \frac{\zeta_0 \cos\iota''}{\nu^2 - a^2} = \frac{1}{A^2}\left\{\frac{\cos\psi\, x'}{a^2} + \frac{\cos\omega\, y'}{a^2 - b^2} + \frac{\cos\phi\, z'}{a^2 - c^2}\right\},$$

of which the geometrical signification is manifest; viz. that if the plane of contact intercept upon any right line (l), drawn from the vertex of the circumscribing cone, a length (l_0), and upon the parallel right line drawn from the centre a length (l_0'), we shall then have

$$\frac{l_0}{l_0'} = A^2.$$

In the equation (7) we have already obtained a particular case of the general relation now demonstrated. Hence, attending to the values found for $(\rho_{,}, \mu_{,}, \nu_{,})$, the right lines intercepted by the plane of contact upon the axes of the circumscribing cone, we deduce

$$\frac{\rho_{,}}{\xi_{,}} = \frac{\mu_{,}}{\eta_{,}} = \frac{\nu_{,}}{\zeta_{,}} = A^2,$$

where $(\xi_{,}, \eta_{,}, \zeta_{,})$ denote the right lines intercepted by the plane of contact upon the perpendiculars (ξ_0, η_0, ζ_0). We consequently obtain the following very general relations:

$$\xi_0 = \frac{1}{\xi_{,}}\,\frac{a^2.(a^2-b^2)(a^2-c^2)}{(a^2-\mu^2)(a^2-\nu^2)},$$

$$\eta_0 = \frac{1}{\eta_{,}}\,\frac{a^2.(a^2-b^2)(a^2-c^2)}{(a^2-\rho^2)(a^2-\nu^2)},$$

$$\zeta_0 = \frac{1}{\zeta_{,}}\,\frac{a^2.(a^2-b^2)(a^2-c^2)}{(a^2-\rho^2)(a^2-\mu^2)}.$$

We now perceive that if at the points in which the axes (ρ, μ, ν) of the circumscribing cone pierce the surface, whose semi-major axis is (a), tangent planes be applied, and upon the perpendiculars (ξ_0, η_0, ζ_0) right lines respectively equal to $(\Xi, \mathrm{H}, \mathrm{Z})$ be intercepted by the tangent planes, then, attending to the values already obtained (*Journal*, vol. v. p. 79) for the intercepted right lines, we shall have

$$\xi_0\xi_{,} = \Xi^2, \quad \eta_0\eta_{,} = \mathrm{H}^2, \quad \zeta_0\zeta_{,} = \mathrm{Z}^2;$$

from which it is easy to infer the conditions under which the perpendiculars (ξ_0, η_0, ζ_0) are divided in continued proportion, according to the variations of the central surface whose semi-major axis is (a). Let us write

$$M^2 = \frac{\cos^2\iota}{\rho^2-a^2} + \frac{\cos^2\iota'}{\mu^2-a^2} + \frac{\cos^2\iota''}{\nu^2-a^2},$$

$$\text{and } N = \frac{\cos\iota\,\xi_0}{\rho^2-a^2} + \frac{\cos\iota'\eta_0}{\mu^2-a^2} + \frac{\cos\iota''\zeta_0}{\nu^2-a^2}:$$

from the equation (6) we shall then have

$$\text{(k)}\quad l^2.(M^2 - A^2N^2) + 2l.A^2N - A^2 = 0.$$

The equation of the surface (S), referred to axes at the point $(x'y'z')$ parallel to the ordinary rectangular axes, is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - b^2} + \frac{z^2}{a^2 - c^2} - 2\left(\frac{xx'}{a^2} + \frac{yy'}{a^2 - b^2} + \frac{zz'}{a^2 - c^2}\right)$$
$$= 1 - \frac{x'^2}{a^2} - \frac{y'^2}{a^2 - b^2} - \frac{z'^2}{a^2 - c^2}.$$

Hence if we write

$$M'^2 = \frac{\cos^2\psi}{a^2} + \frac{\cos^2\omega}{a^2 - b^2} + \frac{\cos^2\phi}{a^2 - c^2}$$

$$\text{and}\quad N' = \frac{\cos\psi x'}{a^2} + \frac{\cos\omega y'}{a^2 - b^2} + \frac{\cos\phi z'}{a^2 - c^2},$$

we shall have (k') $l^2.M'^2 - 2lN' + A^2 = 0$;

consequently $M'^2 = A^2N^2 - M^2$, $N' = A^2N$.

Let (R) be the semi-diameter of the surface (S) parallel to the right line (l), and we shall evidently have

$$(10)\quad \frac{1}{R^2} = \frac{(\rho^2 - a^2)(\mu^2 - a^2)(\nu^2 - a^2)}{a^2(a^2 - b^2)(a^2 - c^2)}$$

$$\left\{\frac{\cos\iota.\xi_0}{\rho^2 - a^2} + \frac{\cos\iota'.\eta_0}{\mu^2 - a^2} + \frac{\cos\iota''.\zeta_0}{\nu^2 - a^2}\right\}^2 - \left(\frac{\cos^2\iota}{\rho^2 - a^2} + \frac{\cos^2\iota'}{\mu^2 - a^2} + \frac{\cos^2\iota''}{\nu^2 - a^2}\right).$$

$N' = A^2N$ is obviously but the equation (9), which we have already obtained. From the equation (10) combined with the equation (a'''), we can readily obtain

$$(11)\quad \left(\frac{1}{a^2} + \frac{1}{a^2 - b^2} + \frac{1}{a^2 - c^2} + \frac{1}{\rho^2 - a^2} + \frac{1}{\mu^2 - a^2} + \frac{1}{\nu^2 - a^2}\right)$$
$$= \frac{(\rho^2 - a^2)(\mu^2 - a^2)(\nu^2 - a^2)}{a^2(a^2 - b^2)(a^2 - c^2)} \cdot \frac{1}{\delta^2}.$$

We shall, however, presently demonstrate that

$$\delta\delta_{,} = a^2 - a_0^2,$$

where (a_0) denotes the semi-major axis of that particular confocal surface which touches the plane of contact. Hence we obtain the interesting relation

$$\frac{1}{a^2} + \frac{1}{a^2 - b^2} + \frac{1}{a^2 - c^2} = \frac{1}{a^2 - a_0^2} + \frac{1}{a^2 - \rho^2} + \frac{1}{a^2 - \mu^2} + \frac{1}{a^2 - \nu^2}:$$

from this equation we can also evidently deduce

$$a^2 - a_0^2 = \delta^2 \frac{a^2.(a^2 - b^2)(a^2 - c^2)}{(\rho^2 - a^2)(\mu^2 - a^2)(\nu^2 - a^2)},$$

or as it may be otherwise written,

$$\frac{(\rho^2 - a^2)(\mu^2 - a^2)(\nu^2 - a^2)}{\delta^2} = \frac{a^2.(a^2 - b^2)(a^2 - c^2)}{a^2 - a_0^2}.$$

But the right-hand member of this equation is, as we shall see, the constant in Joachimsthal's theorem for any geodesic line which touches the line of curvature formed by the intersection of the confocal surfaces (a) and (a_0). From which we infer the following *Theorem:* "Upon any central surface (a) of the second order, let a geodesic line be traced, and where the successive osculating planes along this curve intersect the surface, let cones be circumscribed; then if (ρ, μ, ν) denote the semi-major axes of the confocal surfaces which intersect in the vertex of any one of the enveloping cones, and (δ) the perpendicular from that vertex upon the osculating plane of contact, we shall always have constant the value

$$\frac{(\rho^2 - a^2)(\mu^2 - a^2)(\nu^2 - a^2)}{\delta^2},$$

the constant being Joachimsthal's well-known constant for the geodesic line." Now let (D, D', D'') denote the three rectangular semi-diameters of the surface (S), parallel to the normals (ρ, μ, ν): from the equation (10) we shall have

$$\begin{aligned} &\frac{1}{D^2} = \frac{A^2}{\rho^2 - a^2} \cdot \frac{\xi_0^2}{\rho^2 - a^2} - \frac{1}{\rho^2 - a^2}, \\ (\mathrm{k}'') \quad &\frac{1}{D'^2} = \frac{A^2}{\mu^2 - a^2} \cdot \frac{\eta_0^2}{\mu^2 - a^2} - \frac{1}{\mu^2 - a^2}, \\ &\frac{1}{D''^2} = \frac{A^2}{\nu^2 - a^2} \cdot \frac{\zeta_0^2}{\nu^2 - a^2} - \frac{1}{\nu^2 - a^2}. \end{aligned}$$

Hence, attending to relations already established, we may obtain

$$\rho_, = D^2\left(\frac{1}{\xi_,} - \frac{1}{\xi_0}\right), \quad \mu_, = D'^2\left(\frac{1}{\eta_,} - \frac{1}{\eta_0}\right),$$

$$\nu_, = D''^2\left(\frac{1}{\zeta_,} - \frac{1}{\zeta_0}\right).$$

Conceive the surface (S) to degenerate successively into its three focal curves, and without any difficulty we shall find the following relations connecting the two systems of

coördinates of the vertex of the circumscribing cone:

$$\rho^2\cos\alpha = x'\xi_0, \qquad \mu^2\cos\alpha' = x'\eta_0, \qquad \nu^2\cos\alpha'' = x'\zeta_0,$$
$$(\rho^2-b^2)\cos\beta = y'\xi_0, \quad (\mu^2-b^2)\cos\beta' = y'\eta_0, \quad (\nu^2-b^2)\cos\beta'' = y'\zeta_0,$$
$$(\rho^2-c^2)\cos\gamma = z'\xi_0, \quad (\mu^2-c^2)\cos\gamma' = z'\eta_0, \quad (\nu^2-c^2)\cos\gamma'' = z'\zeta_0,$$

where $(\alpha, \alpha', \alpha'', \beta$, &c.) denote the angles made by the normals (ρ, μ, ν) with the original axes of (x, y, z). From these relations the reader may with facility deduce several not uninteresting properties of central surfaces of the second order.

The solution of the equation (k) gives

$$l = \frac{A^2N \pm AM}{A^2N^2 - M^2};$$

therefore if (l', l'') be the roots of the equation (k), we shall have

$$l' = \frac{A}{AN+M}, \quad l'' = \frac{A}{AN-M}.$$

Hence if (l_1) be the intercept upon the right line (l) made by the surface (S), we shall have

$$l_1 = \frac{2AM}{A^2N^2 - M^2}.$$

Substituting for (A, M, N) their values, we obtain the very remarkable expressions,

$$(12)\quad \left\{\begin{aligned}
l' &= \frac{R^2.\sqrt{(\rho^2-a^2)}.\sqrt{(\mu^2-a^2)}.\sqrt{(\nu^2-a^2)}}{a.\sqrt{(a^2-b^2)}.\sqrt{(a^2-c^2)}} \\
&\qquad \left\{\left(\frac{\cos\iota\,\xi_0}{\rho^2-a^2} + \frac{\cos\iota'\eta_0}{\mu^2-a^2} + \frac{\cos\iota''\zeta_0}{\nu^2-a^2}\right)A \right.\\
&\qquad\qquad \left. + \sqrt{\left(\frac{\cos^2\iota}{\rho^2-a^2} + \frac{\cos^2\iota'}{\mu^2-a^2} + \frac{\cos^2\iota''}{\nu^2-a^2}\right)}\right\}, \\
l'' &= \frac{R^2.\sqrt{(\rho^2-a^2)}.\sqrt{(\mu^2-a^2)}.\sqrt{(\nu^2-a^2)}}{a.\sqrt{(a^2-b^2)}.\sqrt{(a^2-c^2)}} \\
&\qquad \left\{\left(\frac{\cos\iota\,\xi_0}{\rho^2-a^2} + \frac{\cos\iota'\eta_0}{\mu^2-a^2} + \frac{\cos\iota''\zeta_0}{\nu^2-a^2}\right)A \right.\\
&\qquad\qquad \left. - \sqrt{\left(\frac{\cos^2\iota}{\rho^2-a^2} + \frac{\cos^2\iota'}{\mu^2-a^2} + \frac{\cos^2\iota''}{\nu^2-a^2}\right)}\right\}, \\
l_1 &= \frac{2R^2.\sqrt{(\rho^2-a^2)}.\sqrt{(\mu^2-a^2)}.\sqrt{(\nu^2-a^2)}}{a.\sqrt{(a^2-b^2)}.\sqrt{(a^2-c^2)}} \\
&\qquad\qquad \sqrt{\left(\frac{\cos^2\iota}{\rho^2-a^2} + \frac{\cos^2\iota'}{\mu^2-a^2} + \frac{\cos^2\iota''}{\nu^2-a^2}\right)}, \\
l'l'' &= \frac{R^2.(\rho^2-a^2)(\mu^2-a^2)(\nu^2-a^2)}{a^2.(a^2-b^2)(a^2-c^2)}.
\end{aligned}\right.$$

The expression for (l_1) was, I believe, first published by Professor Chasles; but, so far as I am aware, without proof.*

If the relation $\left(\frac{ll'}{R^2}\right)$ be constant, and from its general expression we eliminate (μ) and (ν) by the equations

$$\mu^2 + \nu^2 = x'^2 + y'^2 + z'^2 + b^2 + c^2 - \rho^2,$$

$$\mu^2\nu^2 = \frac{b^2c^2x'^2}{\rho^2},$$

it is easy to see that we shall obtain upon the ellipsoid (ρ), as locus of the vertex of the enveloping cone of the surface (S), a certain curve of the second degree, formed by the intersection of the ellipsoid, with a certain surface of revolution of the second order. The variation of the constant will, of course, enable us to determine the different curves of this species which exist upon the surface of the ellipsoid. Analogous theorems are manifestly true for either of the confocal hyperboloids (μ) or (ν). It is a well-known theorem that, if a perpendicular be let fall from the vertex $(x'y'z')$ of a circumscribing cone upon the plane of contact, its point of intersection with the plane will be the point of contact of that particular confocal surface which touches the plane. Let (a_0) represent the semi-major axis of the latter surface, and (δ_1) the perpendicular from the centre upon the plane of contact; it is easy to see that we shall have the following relations:

$$\frac{\cos \iota}{\delta_1} = \frac{x'}{a^2}, \quad \frac{\cos \iota'}{\delta_1} = \frac{y'}{a^2 - b^2}, \quad \frac{\cos \iota''}{\delta_1} = \frac{z'}{a^2 - c^2},$$

where (ι, ι', ι'') denote the angles which the perpendicular (δ_1) makes with the axes of (x, y, z). Hence it follows, attending to equation (5), that

$$\delta_0^2 - \delta_1^2 = a^2 - a_0^2 = \delta_1^2 A^2,$$

(δ_0) being the perpendicular from the centre upon the plane, parallel to the plane of contact, and tangent to the surface whose semi-major axis is (a). But we have already seen that

$$\frac{\delta}{\delta_1} = A^2 = \frac{L^2}{R^2},$$

(L) being any side of the cone circumscribing the surface (S),

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* Professor Chasles has given a demonstration of this theorem in a very valuable memoir on the Attraction of Surfaces of the Second Order (*Mémoires des Savants Etranges*, tom. IX.): his method is, however, altogether different from that of the present article.

and (R) the parallel semi-diameter; consequently we obtain

$$a^2 - a_0^2 = \delta\delta_1, \quad \delta^2 = (a^2 - a_0^2)\frac{L^2}{R^2}.$$

From which it follows that, if (θ) be the angle made by any side of contact of the circumscribing cone, with the perpendicular from its vertex upon the plane of contact, and (R) the parallel semi-diameter of the circumscribed surface, we shall then have the curious value

$$a^2 - a_0^2 = R^2 \cos^2\theta:$$

the position, therefore, of the plane of contact and any semi-diameter, which is parallel to a side of the enveloping cone, being given, we can determine the semi-major axis of that particular confocal surface which touches the plane of contact. Let (L') denote the right line drawn from the summit of the circumscribing cone to the centre of the circumscribed surface (S), and (R') the coincident radius vector; then, attending to the equations (3) and (k″) and the general value found for

$$\frac{a^2(a^2-b^2)(a^2-c^2)}{(\rho^2-a^2)(\mu^2-a^2)(\nu^2-a^2)},$$

we shall have the very interesting relation

$$\frac{\Xi^2}{D^2}+\frac{\mathrm{H}^2}{D'^2}+\frac{\mathrm{Z}^2}{D''^2}=\frac{L'^2-3R'^2}{L'^2-R'^2},$$

where $(\Xi, \mathrm{H}$, &c., D, D', &c.) represent the same right lines which they have indicated in the preceding portion of the present article.

Let the roots of the equation (k) become equal, the right line (l) will then be a side of the cone which has for vertex the point $(x'y'z')$ and which circumscribes the surface (S). In this case, since M^2 is evidently equal zero, from the equation (10) we shall obtain

$$\frac{1}{R^2}=\frac{(\rho^2-a^2)(\mu^2-a^2)(\nu^2-a^2)}{a^2.(a^2-b^2)(a^2-c^2)}\left\{\frac{\cos\iota.\xi_0}{\rho^2-a^2}+\frac{\cos\iota'.\eta_0}{\mu^2-a^2}+\frac{\cos\iota''.\zeta_0}{\nu^2-a^2}\right\}^2:$$

attending to equation (9), we have therefore

$$\frac{x'\cos\psi}{a^2}+\frac{y'\cos\omega}{a^2-b^2}+\frac{z'\cos\phi}{a^2-c^2}=\pm\frac{1}{R}\sqrt{\left\{\frac{(\rho^2-a^2)(\mu^2-a^2)(\nu^2-a^2)}{a^2.(a^2-b^2).(a^2-c^2)}\right\}};$$

also

$$\frac{\xi_0\cos\iota}{\rho^2-a^2}+\frac{\eta_0\cos\iota'}{\mu^2-a^2}+\frac{\zeta_0\cos\iota''}{\nu^2-a^2}=\pm\frac{1}{R}\sqrt{\left\{\frac{a^2.(a^2-b^2).(a^2-c^2)}{(\rho^2-a^2)(\mu^2-a^2)(\nu^2-a^2)}\right\}}:$$

attending to the equations (12), if (L) be any side of the cone circumscribing the surface (S), we obtain

$$(13)\quad L^2 = \frac{R^2.(\rho^2 - a^2)(\mu^2 - a^2)(\nu^2 - a^2)}{a^2.(a^2 - b^2)(a^2 - c^2)}.$$

But if we conceive the side (L) to touch a second surface, confocal with (S), whose semi-major axis is (a'), I have elsewhere demonstrated (*Journal*, vol. v. p. 85) that

$$(14)\quad P^2L^2 = \frac{(\rho^2 - a^2)(\mu^2 - a^2)(\nu^2 - a^2)}{a^2 - a'^2},$$

where P indicates the perpendicular from the centre of (S) upon the plane tangent to the circumscribing cone along the side (L). Hence we have in its utmost generality the value

$$(15)\quad P^2R^2 = \frac{a^2.(a^2 - b^2)(a^2 - c^2)}{a^2 - a'^2}.$$

Now the plane which passes through the right line (L) and touches the confocal surface whose semi-major axis is (a') will be normal to the surface (S); and since along any geodesic line the osculating plane is normal to the surface, we perceive, without much consideration, the truth of Joachimsthal's theorem that PR = constant for any geodesic line traced upon a central surface of the second order. From the nature of the proof it also follows, that not only is PR = constant for any one geodesic line, but that it is also constant for any group of unique geodesic lines, *i.e.* for all geodesic lines which touch the same line of curvature formed, in the present instance, by the intersection of the surface (S) with the confocal surface whose semi-major axis is (a'). We also see that if tangents be drawn to a central surface of the second order, along a geodesic line or any group of unique geodesic lines, they will be also tangents to a second surface confocal with the first. In general, from any external point four right lines can be drawn common tangents to two confocal surfaces of the second order; consequently, any external point being assumed, we can determine four conjoint elements of geodesic lines, *i.e.* four elements, upon a group of unique geodesic lines, at which if tangents to the surface be drawn they will intersect in a point. It is manifest that these tangents will make equal angles with the normal to the confocal surface which passes through their point of intersection. We may next suppose the external points grouped upon 'loci' of any kind whatsoever; there will, of course, be a corresponding connexion between the conjoint elements determined upon

either surface of the second order, the nature of which it would be most interesting to examine: for the present, however, it must suffice to have indicated the direction in which further investigation may be made.

If the right line (l) be contained in either of the three planes $(\rho\mu)$, $(\rho\nu)$, $(\mu\nu)$, the equations (12) will undergo a corresponding modification. Suppose that it is contained in the plane $(\mu\nu)$, tangent to the confocal surface of which the semi-major axis is (ρ), then

$$l_1 = \frac{2R^2.\sqrt{(\rho^2 - a^2)}}{a.\sqrt{(a^2 - b^2)}.\sqrt{(a^2 - c^2)}} \sqrt{(\mu^2 \sin^2 \iota' + \nu^2 \sin^2 \iota'' - a^2)}.$$

Let us next conceive the right line (l) to touch the surface (S), and we shall then have

$$\mu^2 \sin^2 \iota' + \nu^2 \sin^2 \iota'' = a^2,$$

while at the same time, as is manifest from equation (15), we shall have

$$P^2R^2 = \frac{\rho^2.(\rho^2 - b^2)(\rho^2 - c^2)}{\rho^2 - a^2};$$

we therefore have the known equation of the geodesic line

$$\mu^2 \sin^2 \iota' + \nu^2 \sin^2 \iota'' = \rho^2 - \frac{\rho^2.(\rho^2 - b^2).(\rho^2 - c^2)}{P^2R^2}.$$

This seems to be the most general and direct method of establishing the preceding properties of geodesic lines and lines of curvature, which in fact appear but as particular cases of theorems far more general. From the equation (13) we perceive the curious relation that exists between each side of the circumscribing cone and the parallel diameter of the surface circumscribed; while from the equations (12) and (13) we also obtain the known relation

$$\frac{l'l''}{R^2} = \frac{L^2}{R^2}.$$

From the equation (14), in which the right line (l) is considered as touching the two confocal surfaces whose semi-major axes are (a) and (a'), we can without difficulty deduce the sometimes useful value

$$a^2 - a'^2 = (\rho^2 - a^2)(\mu^2 - a^2)(\nu^2 - a^2)\left\{\frac{\cos^2 \iota}{(\rho^2 - a^2)^2} + \frac{\cos^2 \iota'}{(\mu^2 - a^2)^2} + \frac{\cos^2 \iota''}{(\nu^2 - a^2)^2}\right\}.$$

In a recent number of this *Journal* (vol. IV. p. 81) Dr. Hart has enunciated an important theorem of M. Gauss in the following terms:

"If from any point (O) on a curved surface two geodesic lines be drawn containing the infinitely small angle

$$d\omega = TOT',$$

and if the length $TO = \lambda$, and the perpendicular distance $TT' = Fd\omega$, and if (σ, σ') be the radii of curvature of the surface at (T), then the quantity represented by (F) is such that

$$\frac{d^2F}{d\lambda^2} + \frac{F}{\sigma\sigma'} = 0."$$

When the geodesic lines are traced upon a surface of the second degree, Dr. Hart has obtained with a singularly simple demonstration (vol. IV. p. 192), for the quantity (F), the value

$$F = \frac{R_0}{R} . L.$$

Let (B) be the surface of the second degree, the semi-major axis of which is (ρ), upon which the geodesic line (λ) is traced, touching at (O) the line of curvature formed by the intersection of the confocal surfaces (B) and (S); then if a tangent be drawn to the geodesic line at (T) and prolonged to touch the surface (S) at (Q), L will indicate the length (TQ), R the semi-diameter of (S) parallel to (TQ) and R_0 the semi-diameter of (S) parallel to the tangent to the geodesic line at (O). Now, if (ψ, ω, ϕ) be the angles made by (R_0) with the axes (x, y, z), we shall have

$$\frac{1}{R_0^2} = \frac{\cos^2\psi}{a^2} + \frac{\cos^2\omega}{a^2 - b^2} + \frac{\cos^2\phi}{a^2 - c^2}.$$

If then for $\frac{L}{R}$ we substitute the value found in (13), we shall at length obtain the curious expression

$$(16) \quad F^2 = \frac{(\rho^2 - a^2)(\mu^2 - a^2)(\nu^2 - a^2)}{a^2.(a^2 - b^2)(a^2 - c^2)} \cdot \frac{1}{\frac{\cos^2\psi}{a^2} + \frac{\cos^2\omega}{a^2 - b^2} + \frac{\cos^2\phi}{a^2 - c^2}}:$$

when (S) becomes the focal hyperbola, we shall have the particular value, first demonstrated by Mr. Roberts,

$$F^2 = \frac{(\rho^2 - b^2)(\mu^2 - b^2)(\nu^2 - b^2)}{b^2.(b^2 - c^2)\sin^2\omega},$$

i.e. if (y) denote the ordinate of the point (ρ, μ, ν) normal to the principal plane containing the umbilics $F^2 \sin^2\omega = y^2$.

The preceding are, we believe, the most important *general*

theorems which can be deduced from the equation (6); the *particular* applications are of course numerous, and of various degrees of interest.

2. Let the surface (S) be a paraboloid, the equation of which referred to ordinary rectangular coordinates is

$$\frac{y^2}{p_0} + \frac{z^2}{q_0} = x.$$

Let three confocal paraboloids, the respective equations of which are

$$\frac{y^2}{p} + \frac{z^2}{q} = x + h, \quad \frac{y^2}{p'} + \frac{z^2}{q'} = x + h',$$
$$\frac{y^2}{p''} + \frac{z^2}{q''} = x + h'',$$

intersect in a point (x', y', z'), and at this point let the three normals be drawn. The equation of the paraboloid (S), referred to the three normals as axes, is

$$\left\{\frac{\xi^2}{p-p_0} + \frac{\eta^2}{p'-p_0} + \frac{\zeta^2}{p''-p_0}\right\} \cdot \left\{\frac{\cos^2\alpha}{p-p_0} + \frac{\cos^2\beta}{p'-p_0} + \frac{\cos^2\gamma}{p''-p_0}\right\}$$
$$= \left\{\frac{\xi\cos\alpha}{p-p_0} + \frac{\eta\cos\beta}{p'-p_0} + \frac{\zeta\cos\gamma}{p''-p_0} - \frac{1}{2}\right\}^2,$$

where (α, β, γ) denote the angles made by the axis of (x) with the new rectangular axes (ξ, η, ζ). If we make

$$\frac{\xi^2}{p-p_0} + \frac{\eta^2}{p'-p_0} + \frac{\zeta^2}{p''-p_0} = 0,$$

it will denote the equation of the cone which has its vertex at the point (x', y', z'), and circumscribes the paraboloid (S), then the equation of the plane of contact will be

$$\frac{\xi\cos\alpha}{p-p_0} + \frac{\eta\cos\beta}{p'-p_0} + \frac{\zeta\cos\gamma}{p''-p_0} = \frac{1}{2}:$$

but the equation of this plane in ordinary rectangular coordinates is

$$x + x' = \frac{2yy'}{p_0} + \frac{2zz'}{q_0},$$

which referred to parallel axes of coordinates, intersecting in the point (x', y', z'), becomes

$$(a_{,})\quad x - 2\left(\frac{yy'}{p_0} + \frac{zz'}{q_0}\right) = 2\left(\frac{y'^2}{p_0} + \frac{z'^2}{q_0} - x'\right);$$

consequently, if we write

$$x = \xi \cos\alpha + \eta \cos\beta + \zeta \cos\gamma, \quad y = \xi \cos\alpha' + \eta \cos\beta' + \zeta \cos\gamma',$$
$$z = \xi \cos\alpha'' + \eta \cos\beta'' + \zeta \cos\gamma'',$$

we shall have, writing $A^2 = \dfrac{y'^2}{p_0} + \dfrac{z'^2}{q_0} - x'$,

$$\frac{2\cos\alpha}{p - p_0} = \left\{\cos\alpha - \left(\frac{2\cos\alpha' y'}{p_0} + \frac{2\cos\alpha'' z'}{q_0}\right)\right\} \cdot \frac{1}{2A^2},$$
$$\frac{2\cos\beta}{p' - p_0} = \left\{\cos\beta - \left(\frac{2\cos\beta' y'}{p_0} + \frac{2\cos\beta'' z'}{q_0}\right)\right\} \cdot \frac{1}{2A^2},$$
$$\frac{2\cos\gamma}{p'' - p_0} = \left\{\cos\gamma - \left(\frac{2\cos\gamma' y'}{p_0} + \frac{2\cos\gamma'' z'}{q_0}\right)\right\} \cdot \frac{1}{2A^2}.$$

From which, without any difficulty, we obtain

$$\frac{\cos^2\alpha}{p - p_0} + \frac{\cos^2\beta}{p' - p_0} + \frac{\cos^2\gamma}{p'' - p_0} = \frac{1}{4\left(\dfrac{y'^2}{p_0} + \dfrac{z'^2}{q_0} - x'\right)};$$

therefore the equation of the paraboloid (S) becomes

$$\frac{\xi^2}{p - p_0} + \frac{\eta^2}{p' - p_0} + \frac{\zeta^2}{p'' - p_0}$$
$$= 4\left(\frac{y'^2}{p_0} + \frac{z'^2}{q_0} - x'\right)\left\{\frac{\xi\cos\alpha}{p - p_0} + \frac{\eta\cos\beta}{p' - p_0} + \frac{\zeta\cos\gamma}{p'' - p_0} - \frac{1}{2}\right\}^2:$$

but since three confocal paraboloids intersect in the point (x', y', z'), we can prove that

$$x' = \frac{(p_0 - p') + (p_0 - p'') - (p + q_0)}{4}, \quad y'^2 = \frac{pp'p''}{4(q_0 - p_0)}$$
$$z'^2 = \frac{qq'q''}{4(p_0 - q_0)}.$$

Hence after some obvious reductions we obtain

$$\frac{y'^2}{p_0} + \frac{z'^2}{q_0} - x' = \frac{(p - p_0)(p' - p_0)(p'' - p_0)}{4p_0q_0},$$

we consequently have as the ultimate form of the equation of the paraboloid (S),

$$(17) \quad \frac{\xi^2}{p - p_0} + \frac{\eta^2}{p' - p_0} + \frac{\zeta^2}{p'' - p_0}$$
$$= \frac{(p - p_0)(p' - p_0)(p'' - p_0)}{p_0q_0}\left\{\frac{\xi\cos\alpha}{p - p_0} + \frac{\eta\cos\beta}{p' - p_0} + \frac{\zeta\cos\gamma}{p'' - p_0} - \right\}^2.$$

The equation of the plane of contact (a,) also becomes

$$(\text{a}_,') \quad x - 2\left(\frac{yy'}{p_0} + \frac{zz'}{q_0}\right) = \frac{(p-p_0)(p'-p_0)(p''-p_0)}{2p_0q_0}.$$

If (δ) denote the perpendicular upon this plane from the point (x', y', z'), we shall have

$$\frac{1}{\delta^2} = \frac{1}{4}\left(\frac{\cos^2\alpha}{p-p_0} + \frac{\cos^2\beta}{p'-p_0} + \frac{\cos^2\gamma}{p''-p_0}\right),$$

and

$$\delta = \frac{\frac{1}{2}\cdot\frac{(p-p_0)(p'-p_0)(p''-p_0)}{p_0q_0}}{\sqrt{\left\{1 + 4\left(\frac{y'^2}{p_0^2} + \frac{z'^2}{q_0^2}\right)\right\}}}:$$

consequently we are now enabled to extend in its modified form the theorem (8), that we have already obtained for the central surfaces, to the non-central surfaces, viz.

$$\frac{\frac{y'^2}{p_0} + \frac{z'^2}{q_0} - x'}{\sqrt{\left\{1 + 4\left(\frac{y'^2}{p_0^2} + \frac{z'^2}{q_0^2}\right)\right\}}} = \frac{\delta}{2}.$$

If we now put $M^2 = \frac{\cos^2\iota}{p-p_0} + \frac{\cos^2\iota'}{p'-p_0} + \frac{\cos^2\iota''}{p''-p_0}$ and

$$N = \frac{\cos\iota.\cos\alpha}{p-p_0} + \frac{\cos\iota'.\cos\beta}{p'-p_0} + \frac{\cos\iota''.\cos\gamma}{p''-p_0},$$

we shall have

$$(\text{k}_,) \quad l^2.(M^2 - 4A^2N^2) + l.4A^2N - A^2 = 0:$$

but the equation of the paraboloid (S), referred to ordinary rectangular coordinates, the axes of which intersect in the point (x', y', z'), is

$$\frac{y^2}{p_0} + \frac{z^2}{q_0} + 2\left(\frac{yy'}{p_0} + \frac{zz'}{q_0} - \frac{x}{2}\right) = x' - \frac{y'^2}{p_0} - \frac{z'^2}{q_0};$$

consequently, if we put $M'^2 = \frac{\cos^2\omega}{p_0} + \frac{\cos^2\phi}{q_0}$ and

$$N' = \frac{\cos\omega\, y'}{p_0} + \frac{\cos\phi.z'}{q_0} - \frac{\cos\psi}{2},$$

we shall have $(\text{k}_,') \quad l^2M'^2 + 2lN' + A^2 = 0;$

where (l) indicates any radius vector of the paraboloid (S) drawn from the origin of coordinates (x', y', z') and (ι, ι', ι''),

the angles which it makes with the axes (ξ, η, ζ) and (ψ, ω, ϕ) the angles which it makes with the axes (x, y, z).

Since $M'^2 = 4A^2N^2 - M^2$ and $\frac{\cos^2\omega}{p_0} + \frac{\cos^2\psi}{q_0} = \frac{1}{\chi}$ where (χ) indicates the length of the bifocal chord of the paraboloid (S) parallel to the radius vector (l), we have

$$\frac{1}{\chi} = \frac{(p-p_0)(p'-p_0)(p''-p_0)}{p_0q_0}\left(\frac{\cos\iota.\cos\alpha}{p-p_0} + \frac{\cos\iota'.\cos\beta}{p'-p_0} + \frac{\cos\iota''.\cos\gamma}{p''-p_0}\right)^2$$
$$-\left(\frac{\cos^2\iota}{p-p_0} + \frac{\cos^2\iota'}{p'-p_0} + \frac{\cos^2\iota''}{p''-p_0}\right).$$

Now if we denote by (l_i) the difference of the roots (l', l'') of the equation (k_i), we shall have

$$(18)\quad l_i = \chi\frac{\sqrt{(p-p_0)}\sqrt{(p'-p_0)}\sqrt{(p''-p_0)}}{2\sqrt{(p_0q_0)}}\sqrt{\left(\frac{\cos^2\iota}{p-p_0} + \frac{\cos^2\iota'}{p'-p_0} + \frac{\cos^2\iota'}{p''-p_0}\right)}$$

and
$$l'l'' = \chi\frac{(p-p_0)(p'-p_0)(p''-p_0)}{4p_0q_0}.$$

Let the roots (l', l'') of the equation (k_i) be equal, the right line (l) will then be a side of the cone which has its vertex at the point (x', y', z') and circumscribes the paraboloid (S). Let (L) indicate the length of any side of this cone, and we shall have

$$L^2 = \chi\cdot\frac{(p-p_0)(p'-p_0)(p''-p_0)}{4p_0q_0},$$

we consequently perceive the pleasing relation that exists between the side of the circumscribing cone and the parallel bifocal chord of the paraboloid circumscribed; we have also in general the known relation

$$\frac{l'l''}{\chi} = \frac{L^2}{\chi}.$$

Let (P) denote the perpendicular from the summit of the paraboloid upon any tangent plane to the circumscribing cone, and let (L) denote the length of the side of contact; we can readily prove that

$$\frac{4x^2}{P^2L^2} = \frac{\cos^2\iota}{(p-p_0)^2} + \frac{\cos^2\iota'}{(p'-p_0)^2} + \frac{\cos^2\iota''}{(p''-p_0)^2},$$

where (x) denotes an ordinate of the point of contact of the tangent plane with the paraboloid. Let us next conceive that the side (L) touches also the paraboloid, the parameter

of which in the principal plane (y, x) is $p_{,}$; we have elsewhere demonstrated (*Journal*, vol. v. p. 90) that the values of $(\cos\iota, \cos\iota', \cos\iota'')$ then are

$$\cos^2\iota = \frac{(p-p_0)(p-p_{,})}{(p-p')(p-p'')}, \quad \cos^2\iota' = \frac{(p'-p_0)(p'-p_{,})}{(p'-p)(p'-p'')},$$

$$\cos^2\iota'' = \frac{(p''-p_0)(p''-p_{,})}{(p''-p)(p''-p')}:$$

substitute these values in the preceding expression, and we shall have

$$\frac{P^2L^2}{x^2} = \frac{(p-p_0)(p'-p_0)(p''-p_0)}{4(p_0-p_{,})}.$$

Hence, attending to the value already found for (L^2), we shall have

$$\frac{P^2\chi}{x^2} = \frac{p_0q_0}{p_0-p_{,}};$$

we have therefore Joachimsthal's theorem for geodesic lines and lines of curvature on *central* surfaces of the second order, now stated for non-central surfaces of the second order: and since, in general, from any external point four right lines can be drawn common tangents to two confocal surfaces, there are therefore, in general, four conjoint elements upon the paraboloid (S), determined by every external point (x', y', z'), for which $\frac{P^2\chi}{x^2} = \text{constant}$. We may here remark that the conclusions at which we have arrived in the analogous case for the central surfaces of the second order, together with the many very beautiful theorems stated by different authors concerning geodesic lines and lines of curvature, may now without difficulty be stated in their modified form for the non-central surfaces: on this part of our subject it is needless, however, to dwell longer. Let the right (l) be contained in the plane $(\eta\zeta)$, and the equation (18) will become

$$l_{,} = \frac{\chi\sqrt{(p-p_0)}}{\sqrt{(p_0q_0)}}.\sqrt{(p''\sin^2\iota'' + p'\sin^2\iota' - p_0)}.$$

Let us next suppose that it touches the paraboloid (S), we shall then have

$$p''\sin^2\iota'' + p'\sin^2\iota' = p_0,$$

while at the same time we manifestly have

$$\frac{P^2\chi}{x^2} = \frac{pq}{p-p_0}:$$

we consequently obtain for the geodesic line upon the paraboloid, the parameters of which in the planes of (yx) and (zx) are (p) and (q), the equation

$$p'' \sin^2 \iota'' + p' \sin^2 \iota' = p - \frac{x^2 pq}{P^2 \chi}.$$

(P) of course indicates in the present instance the perpendicular from the origin of the ordinary rectangular coordinates upon the tangent plane to the last-named paraboloid at any point upon the geodesic line; (χ) the bifocal chord of the same paraboloid parallel to the geodesic tangent at the point of contact of the tangent plane, and (x) an ordinate of that point. When the right line (l) touches the two confocal paraboloids, the parameters of which in the principal plane (yx) are (p_0) and (p_i), as in the analogous case of the central surfaces, we can prove that

$$(p_0 - p_i) = (p - p_0)(p' - p_0)(p'' - p_0) \cdot \left\{ \frac{\cos^2 \iota}{(p-p_0)^2} + \frac{\cos^2 \iota'}{(p'-p_0)^2} + \frac{\cos^2 \iota''}{(p''-p_0)^2} \right\}.$$

From the principles which have been enumerated, we can without any difficulty perceive that the equation (16) becomes when modified for the paraboloids,

$$F^2 = \frac{\chi_0}{\chi} . L^2,$$

where (χ) denotes the bifocal chord of the paraboloid (S) parallel to the tangent to the geodesic line at the point (T) upon the paraboloid (B), and (χ_0) the bifocal chord of the paraboloid (S) parallel to the tangent to the geodesic line at the point (O): $(B, O,$ and $T)$ in the present instance indicate a surface, and two points analogous to those indicated in the former instance for the central surfaces. Substitute for $\frac{L^2}{\chi}$ and χ_0 their values, and we shall have

$$F^2 = \frac{(p - p_0)(p' - p_0)(p'' - p_0)}{4 p_0 q_0} \cdot \frac{1}{\dfrac{\cos^2 \omega}{p_0} + \dfrac{\cos^2 \phi}{q_0}}.$$

3. When the surface (S) is a central surface, as in section (1), the reciprocal surface with respect to a sphere of radius unity, the centre of which is at the point (x', y', z'), has for its equation, in coordinates (ξ, η, ζ),

$$(\rho^2 - a^2)\xi^2 + (\mu^2 - a^2)\eta^2 + (\nu^2 - a^2)\zeta^2 = 2(\xi\xi_0 + \eta\eta_0 + \zeta\zeta_0) - 1,$$

where (ξ_0, η_0, ζ_0) indicate the coordinates of the centre of (S). This equation was first given by MacCullagh without proof; the following is, however, perhaps the most simple possible. Let (S') be the reciprocal surface, and at any point let a tangent plane be applied; if (r) be the vector of this point drawn from the centre (x', y', z') of the reciprocating sphere, then perpendicular to (r) produced, let a plane be drawn tangent to the surface (S), and calling (p) the length of the perpendicular intercept, we shall have from the nature of the reciprocity $pr = 1$. If a parallel plane pass through the centre of (S), and δ be the length of the perpendicular upon this plane from (x', y', z'), and $(\delta_{,})$ the length of the perpendicular from the centre of (S) upon its tangent plane, normal to (δ), *i.e.* normal to the produced vector (r), we shall have $p = \delta - \delta_{,}$; therefore

$$r\delta_{,} = \delta r - 1,$$

$$(\mathrm{k}_{,,}) \quad r^2 (\delta^2 - \delta_{,}^2) = 2\delta r - 1.$$

Now if $\xi.\cos\lambda + \eta.\cos\varpi + \zeta.\cos\tau = 0$ be the equation of the plane normal to (δ) which passes through (x', y', z'), we shall have $\delta = \xi_0 \cos\lambda + \eta_0 \cos\varpi + \zeta_0 \cos\tau$; and if (a') be the semi-major axis of the surface confocal with (S), which touches the last-mentioned plane, we shall have

$$a'^2 - a^2 = \delta^2 - \delta_{,}^2.$$

It is also a well-known theorem that

$$a'^2 = \rho^2 \cos^2\lambda + \mu^2 \cos^2\varpi + \nu^2 \cos^2\tau,$$

where (ρ, μ, ν) denote the semi-major axes of the three surfaces confocal with (S), which intersect in the point (x', y', z'), and (λ, ϖ, τ) the angles which the three normals at (x', y', z') make with δ; substitute these values in the equation $(\mathrm{k}_{,,})$, and we shall have

$$(\rho^2 - a^2)\,\xi^2 + (\mu^2 - a^2)\,\eta^2 + (\nu^2 - a^2)\,\zeta^2 = 2\,(\xi\xi_0 + \eta\eta_0 + \zeta\zeta_0) - 1.$$

The asymptotic cone of this surface is evidently reciprocal to the cone which has its vertex at the point (x', y', z'), and circumscribes the surface (S). If we reciprocate the preceding equation with respect to a sphere of radius unity, we shall obtain the equation (1). This is perhaps the most easy, although an indirect method of obtaining that expression. In the present instance however, as is indeed in general the case, the direct method by the ordinary transformation of coordinates possesses an important advantage, inasmuch as it indicates several interesting relations, which perhaps would otherwise remain unnoticed; or if noticed,

would occur as so many isolated properties, without reference to their position in the theory of surfaces of the second order. When the surface (S) is a paraboloid, as in section (2), this equation becomes, as MacCullagh remarked,

$$(p-p_0)\xi^2+(p'-p_0)\eta^2+(p''-p_0)\zeta^2=4(\xi\cos a+\eta\cos\beta+\zeta\cos\gamma),$$

where (a, β, γ) are the angles made by the normals at the point (x', y', z') with the axis of (x). It is manifest that the circular sections of the first of the two preceding surfaces are normal to the right lines

$$\frac{\xi^2}{\rho^2-\mu^2}+\frac{\zeta^2}{\nu^2-\mu^2}=0;$$

its circular sections are therefore normal to the generatrices of the hyperboloid of one sheet confocal with (S) which passes through the point (x', y', z'). Similarly the circular sections of the second surface, which are evidently normal to the right lines represented by the equation

$$\frac{\xi^2}{p-p'}+\frac{\zeta^2}{p''-p'}=0,$$

are normal to the generatrices of the hyperbolic paraboloid, confocal with the parabolid (S), and passing through the point (x', y', z'). These very beautiful theorems were first demonstrated by MacCullagh.

September 18, 1849.

PROBLEM RESPECTING POLYGONS IN A PLANE.

By Robert Moon.

[To the Editor of the *Cambridge and Dublin Mathematical Journal.*]

The following results which occurred to me while considering a problem suggested by my friend Mr. Sylvester, may possibly interest some of your readers. The analysis which led to them is so simple that I deem it unnecessary to set it down here.

If we draw lines from the extremities of the five diagonals of any pentagon in a plane to any other point in the plane, either within or without the figure, the sum of the five triangles so formed, each taken with its proper sign according to a certain rule presently to be explained, will be equal to

the five-rayed star bounded by the dark line (fig. 1), *plus* the internal pentagon formed by joining each *consecutive* pair of the internal angles of the star.

(1)

Also, if we have any polygon of n sides, where n is odd and greater than three, and diagonals be drawn by joining the angular points of the polygon *alternately*, the n triangles which have the diagonals so drawn for their bases, and which have also a common vertex at any other point of the plane, when taken together, with their proper signs, will be equal to a star of n rays formed similarly to the five-rayed star in the last case, *plus* the internal figure of n sides formed by joining the internal angles of the star taken *consecutively*.

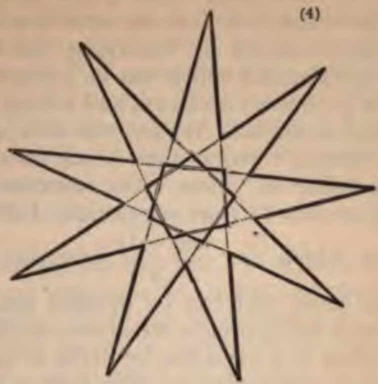
(2)

If we have any heptagon in a plane, and join the points which occur *at intervals of two* instead of alternately, and form triangles by drawing lines from any point in the plane to the extremities of the diagonals so drawn, the sum of such triangles, each with its proper sign, will be equal to the seven-rayed star delineated in (3) by a black line, *plus* a second internal seven-rayed star, formed by joining the internal angles of the first star taken *alternately* (which second star is delineated in the same figure by the shaded line), *plus* the heptagon formed by joining the internal angles of the interior star taken *consecutively*. The same problem may be extended to any polygon of an odd number of sides in which diagonals can be drawn, leaving two or more angular points of the figure between each of their extremities, on whichever side of the diagonal the points be counted.

(3)

If we have a nonagon, and join each pair of points which occur at *intervals of three*, the sum of the corresponding triangles will be = (1) A nine-rayed star formed by the diagonals so drawn + (2) An internal similar star formed by joining the internal angles of the first *taken at intervals of*

[t]o + (3) A third similar star formed by joining the internal [a]ngles of the second, taken [a]*lternately* + (4) A nonagon [f]ormed by joining the in[t]ernal angles of the third star taken *consecutively*.



Further extensions of the problem will readily suggest themselves.

It remains to explain the rule of signs.

Let the angles of the polygon taken in order be numbered 1.2.3.4.5.

Let F be the vertex, and let the series of triangles be written

$$(a) \quad F(1, 3) + F(2, 4) + F(3, 5) + F(4, 1) + F(5, 2),$$

where $F(1, 3)$ denotes the triangle whose base is the line joining the angles 1, 3, &c. Then if all those triangles for which the radius vector from F in sweeping from the first of the angles designated in the expression for each triangle to the second moves to the right, be considered positive, those for which the radius vector under the same circumstances moves to the left must be considered negative. Thus the only requisite for finding the signs is to write down properly the series corresponding to (a), the law of which may be readily seen from inspection. As a further illustration of it, I shall write down the series of triangles corresponding to the case figured in (3), omitting the letter F which occurs in the expression for each,

$$(1, 4) + (2, 5) + (3, 6) + (4, 7) + (5, 1) + (6, 2) + (7, 3).$$

6, New Square, Lincoln's Inn, May 14, 1849.

[*Addition, dated Nov.* 1, 1849, *and addressed to the Editor.*]

In reference to the passage in your note of the 19th ult. in which you state that you had written concerning my paper to Professor De Morgan, and that he had "suggested the construction of such figures from polygons of even numbers of sides," and that he had in his letter drawn you "one from an octagon," I beg to say that I never had any doubt of the possibility of constructing such figures in the case of polygons of an even number of sides, but I did doubt as to

whether the proposition as to the sums of the areas held in this latter case. The doubt which I had in this case arose from the fact that my attention had first been called to the above results by observing that the sum of the areas of all the triangles which can be formed by joining any odd number of points two and two, and taking the lines so drawn for bases and a common vertex, was independent of the position of the vertex; a proposition which does not hold respecting an *even* number of points. On re-examining the subject, however, pursuant to your suggestion, I observe that, in the same way in which the $\frac{n.(n-1)}{1.2}$ triangles which occur in the case of n being odd can be divided into a certain number of sets, each containing n members, so also the triangles which occur when n is even can be divided into a certain number of sets each containing n, *plus* half a set containing $\frac{1}{2}n$ members. Thus, if we have six points, the (algebraical) sum of the six triangles which have any common vertex, and for their bases the lines joining the six points taken *alternately*, will be a six-rayed star, *plus* an internal hexagon. But if we join the six points *taken at intervals of two*, we shall in this way have only *three* lines for bases, and consequently only *three* triangles. It will also be seen that the three bases in this case do not make up an inclosed space, and the sum of the corresponding triangles is *not* independent of the position of the vertex.

If we have eight points and join them two and two, taking them *alternately*, we shall in like manner have the sum of the corresponding triangles equal to an eight-rayed star, *plus* an internal octagon. If we join the points taken at intervals of two, the sum of the corresponding triangles will be equal to an eight-rayed star, *plus* an internal similar star, *plus* an octagon inscribed within the latter. If we join the eight points taken at intervals of *three*, we shall only have four triangles, whose sum will depend on the position of their vertex, and whose bases do not make up an inclosed space. Thus generally, whether n be odd or even, if we join each pair of n points which occur at intervals of m, the sum of the triangles having any common vertex and the lines so drawn for their bases will be equal to m stars of n rays, inscribed each within the other, *plus* an n-sided figure inscribed within the innermost star, it being understood that m cannot exceed $\frac{1}{2}(n-3)$ when n is odd, or $\frac{1}{2}(n-4)$ when n is even.

In conclusion, I shall take the opportunity of stating a proposition naturally connected with this subject.

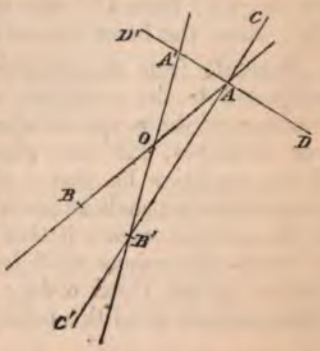
If we have any number of lines of definite, though various, lengths, we may always draw through a given point a locus-line such that the algebraical sum of the areas of all the triangles which can be formed by taking the given lines for bases, and any point in the locus *line for the common vertex, shall be invariable; subject to the following rule of signs, that when the* locus *line crosses a base line or base line produced, if in estimating such sum we assume the area* corresponding to that base *to be positive when the vertex is on the one side of the base line, it must be taken to be negative when the vertex is on the other side of the same line.*

It should be observed that, after the base lines and the point through which the *locus* line is to be drawn are given, it is not necessary, in estimating the "sum of the areas of all the triangles," that we should assign to the latter the same sign, nor does the sign of one depend upon those of the others according to any rule, but either a positive or negative sign may be assigned to each at discretion.

From the above it follows that:

The locus of a point, the sum of the perpendiculars from which upon any given number of straight lines is invariable, is a straight line; where by 'sum of the perpendiculars' is meant the algebraical sum, and it being understood that in comparing the sums of the perpendiculars from two points on opposite sides of any base line, the perpendiculars upon that line *from those points shall be considered to have opposite signs.*

As a simple illustration of both propositions I give the following: AB, $A'B'$ are two lines intersecting in O. Take $OA = OB'$, $OA' = OA$, and draw $CAB'C'$, $DAA'D'$. The *sum* of the perpendiculars from any point of $CAB'C'$ situated between A and B' upon AB and $A'B'$ is invariable, and equal to the *difference* between the perpendiculars from any other point of $CAB'C'$ upon the same lines. On the other hand, the *sum* of the perpendiculars from any point of DD' between A and A' will be invariable and equal to the difference between the perpendiculars from any other point of DD' on the same lines. From which it will be gathered that from any point



within the angle AOB' a line may be drawn such that either the sum or difference of the perpendiculars from any point in that part of it confined within that angle upon the two given lines will be invariable; and similarly of the angles AOA', $A'OB$, and BOB'.

The former of the above propositions seems to be a proper generalization of Euclid's theorem "that triangles on the same base and between the same parallels are equal."

If we have any number of surfaces of definite extent, either plane or curved, and we form pyramids by drawing radii from any common vertex to the bounding edges of the several surfaces; the *locus* of the common vertex when subjected to the condition of rendering invariable the algebraical sum of the volumes of all the pyramids taken with any arbitrary predetermined signs, is a plane; it being understood that, in comparing the algebraical sums of the volumes, if when the vertex is on one side of any *plane* base the sign of the pyramid corresponding to that base is taken to be positive, the sign of the same volume when the vertex is on the opposite side of the plane must be assumed to be negative. The rule of signs in the case of any curved surface being a base, will be determined by the consideration that the sign of the volume corresponding to any indefinitely small curved area will be invariable so long as the common vertex remains on the same side of *the tangent plane* at the point.

It will easily be seen how, by properly adapting the last remark, the proposition previously stated with regard to the sum of the areas of any number of triangles in a plane may be extended from the case where the bases are rectilinear to that where the bases are curvilinear.

It results from the foregoing that the locus of a point when subjected to the condition of rendering invariable a linear function of the perpendiculars drawn from it upon any number of planes, will be a plane. It may also be shewn that if the condition be that a function of *two* dimensions of the perpendiculars shall be invariable, the locus will be a surface of the second order; if that a function of the perpendiculars of three dimensions shall be invariable, the locus will be a surface of the third order; and so on. The same rule of signs prevails as in the preceding cases.

NOTES ON THE PRECEDING PAPER.

Note A. *See* p. 135, line 5.

[The "locus line," passing through a given point O, may be determined in the following manner.

Let L_1L_1', L_2L_2',...L_nL_n' be the given lines (a diagram is unnecessary), lettered in such a manner that if a person looks from L_1 towards L_1', the side of the line L_1L_1' on which triangles having it for base are (in Mr. Moon's rule of signs) reckoned as positive will be to the left hand; and so for the other lines. And farther, for the sake of avoiding unnecessary complexity with reference to estimates of the "areas" to be mentioned below, let us suppose that the lines are numbered in order by the suffixes, so that a line revolving in the plane in a contrary direction to that of the motion of the hands of a watch shall be successively parallel to L_1L_1', L_2L_2', L_3L_3', and towards the same parts. Through the given point O draw OA_1 equal and parallel to L_1L_1', and towards the same parts; through A_1 draw A_1A_2 equal and parallel to L_2L_2'; and so on till a line $A_{n-1}A_n$ is drawn parallel to L_nL_n'. Join OA_n: this line is the required locus.

For the area of the polygon $OA_1A_2...A_n$ together with half the algebraic sum of the parallelograms $OL_1L_1'A_1$, $A_1L_2L_2'A_2$,... and $A_{n-1}L_nL_n'A_n$, is equal to half the algebraic sum of the products of the given lines into perpendiculars drawn to them from any point of OA_n; or to the algebraic sum of the areas of the triangles which have for their bases the given lines, and any point in OA_n for their common vertex; and hence, wherever the point be in the line OA_n (or in this line produced in either direction) the sum of the areas of these triangles is the same. This property will be possessed by any line through the point O, in the special case of the data being such that A_n coincides with O; and therefore in this case, wherever the common vertex be situated in the plane of the given lines, the sum of the triangles will be the same. The condition for this case will be satisfied if the given lines be arranged so as to form a closed circuit of any kind, whether an ordinary polygon, a polygon with re-entrant angles, or a "self-cutting" circuit; provided the circuit may be traced by proceeding along the different lines with the "positive side" always on the left, or, according to the notation of this note, from the non-accented to the accented terminal letter of each of the given lines; and it is clear that *the constant sum of the areas of the triangles is equal to the area of the closed circuit;* the term 'area' being used in the extended sense explained by Professor De Morgan in the paper which follows, if the circuit be "self-cutting." The propositions given by Mr. Moon at the commencement of his paper are all included in this enunciation.

If the data be such that, with the construction which has been described, the point A_n does not coincide with O, it is obvious that the locus, through any other point O', of the common vertex of the triangles when the sum of the areas is constant, is a straight line parallel to OA_n; and that this constant value will exceed the sum of the areas when the common vertex is in OA_n, or fall short of it, by half the area of a parallelogram on OA_n as base and between it and the parallel through O', according as this parallel is on the same side of OA_n as the polygon $OA_1A_2......A_n$, or on the opposite side. The locus of the vertex of the triangles when the sum of their areas is zero may consequently be found by drawing a parallel to OA_n at a distance from it equal to twice the sum of the areas when the vertex is in OA_n, divided by the length of OA_n, on the same side as

the polygon $OA_1A_2...A_n$, when this sum is negative, and on the opposite side when it is positive.

If the given lines represent in magnitude and position a system of forces acting in one plane on a rigid body, in directions from L_1 towards L_1', from L_2 towards L_2', &c. respectively, the construction for finding the resultant of all the forces, transferred to a point O by the introduction of couples, is precisely the construction given above for determining the line OA_n. Hence it appears that a force acting from O towards A_n, and represented in magnitude by OA_n, and the resultant of the couples of transference, which will be a couple in the plane of the forces, with a moment equal to twice the sum of the triangles having O for vertex, and, for their bases, the lines representing the given forces, are together equivalent to the given system. Hence Mr. Moon's proposition, when applied to the mechanical system, expresses that the moment of the resultant of the couples of transference is the same, to whatever point in the line of action of the resultant force through O the forces be transferred. In the special case when A_n coincides with O, the resultant force vanishes. The geometrical proposition which has been proved above, expresses for this case that the moment of the resultant couple of transference (which will be the complete resultant of the given system of forces) is the same whatever be the point in their plane of action to which the forces are transferred. If the lines representing the forces constitute a closed circuit, which may be traced by proceeding along each line in the direction of the force which it represents, it appears, from the geometrical proposition stated above, that the system is equivalent to a resultant couple of which the moment is equal to twice the area of the circuit. Lastly, when the data are such that A_n does not coincide with O, the whole system will be equivalent to a single force, equal, parallel, and in a similar direction to that of the force represented by OA_n, and acting in the line determined above as the locus of the common vertex of the triangles, when the sum of their areas is zero.

These mechanical propositions and the constructions relating to them are all well known; but the brief notice of them which has been introduced may be not uninteresting, in connexion with Mr. Moon's propositions and diagrams. It may also be remarked that the determination of the resultant couple of a system of forces represented by lines forming a closed self-cutting circuit, affords an interesting application of Professor De Morgan's rule for the extended interpretation of the term 'area,' the half-moment of this couple being equal (numerically if the lines have the same numerical values as the forces which they represent) to the area, in the extended sense of the term of the circuit.

Note B. *See* p. 136, l. 12.

It is easily proved that the locus planes, corresponding to different values of the sum of the volumes of the pyramids, are parallel to one another; and a geometrical construction may be readily given to find that one of the series for which the value of the sum is zero, unless the data be such that the sum of the projections of all the given areas on any plane is zero, when, as is easily shewn, the sum of the volumes of the pyramids is the same whatever be the position of their common vertex in space. The condition of exception is fulfilled in the following cases.

(1). When the given areas constitute an unbroken surface (whether polyhedral or curved) which completely encloses a single portion of space; provided that either their positive sides or their negative sides are all towards this interior space.

(2). When the given areas constitute a self-cutting surface which is *complete*, that is *without holes;* provided that round the boundary of each of them, the positive sides of the contiguous areas are towards the same portion of space as its own positive side.

Hence in both these cases the sum of the volumes of the pyramids which have the given areas for bases, and any point in space for their common vertex, is constant. In the first case the value of this sum is the volume of the space enclosed (reckoned of course as positive or negative according as the positive or negative sides of the given areas are towards the interior); and, by an extension of the term *volume* similar to the "extension of the term area" explained by Professor De Morgan in the paper which follows this, the constant value of the sum of the pyramids in the second case may be called the *volume* of the given *complete auto-tomic surface.*—W. T.

Glasgow College, March 1, 1850.]

EXTENSION OF THE WORD *AREA.*

By Professor De Morgan.

I was led to the following investigation by observing the manner in which it is usually made to appear that the area of a plane rectilinear figure of n sides, whose m^{th} angular point has the coordinates x_m, y_m, is

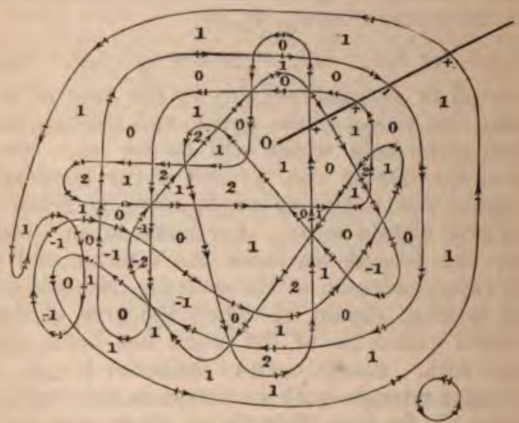
$$\pm\tfrac{1}{2}\{x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + \ldots. + x_ny_1 - x_1y_n\}.$$

It is usual to demonstrate this in the case of a triangle by drawing one phase of the figure, and trusting to algebra to produce a formula which will do for all. No account is given of the double sign $\pm$: and if the boundary of the figure should cut itself, no explanation is given of what the word *area* should mean. For instance, in a star-shaped pentagon, the formula includes the interior and ordinary pentagon *twice*, and each of the outlying triangles *once;* and this total is what should be called the *area* of the figure: but no such extension of the word has been made, that I ever met with. Lastly, each of the terms $\frac{1}{2}(x_my_{m+1} - x_{m+1}y_m)$ representing a triangle with its vertex in the origin and one of the sides for its base, there is no demonstration that the formula gives the same result wherever the origin may be: but this is obvious enough as to an ordinary area.

If the sides become infinitely small, each of the elements takes the form $\frac{1}{2}r^2d\theta$, positive or negative with $d\theta$: and the whole area is the integral $\frac{1}{2}\int r^2d\theta$, in what may be called an *alternating* form: that is, considering $\frac{1}{2}\int r^2d\theta$ as the algebraic sum of the *ordinary* integrals from $r = \alpha$ to $r = \alpha + \beta$, and thence to $r = \alpha + \beta - \gamma$, and thence to $r = \alpha + \beta - \gamma + \delta$, and

so on. A polygon of any kind is included if discontinuous expressions for r be admitted. Given a line which returns into itself, no matter how *autotomic* its character, and having given a point, internal or external, as the pole of coordinates, I shall assume as the *area contained within that circuit* the area swept over by the radius vector which has one end in the pole and the other describing the circuit,* on the supposition that each element is positive or negative, according as the radius is revolving positively or negatively. Of this definition it is to be shown—that it satisfies existing notions (and this I may leave the reader to shew for himself) —that it provides the necessary extensions—and that it gives to every circuit the same area, whatever point the pole of coordinates may be. The double sign of the whole is at once explained: there are two ways of making the circuit; and in passing from one mode to the other, every element changes sign.

Let any straight line be drawn, beginning at the pole of revolution; and let the positive and negative directions of revolution be settled, as also the mode of describing the



circuit. Let this last be done, and every time the circuit crosses the straight line let the sign of the revolution be

* It is to be remarked that a curve which has double or multiple points may be in many different ways a *circuit*, or mode of proceeding from one point to the same again. Thus a figure of 8 may be traced as a *self-cutting* circuit, in the way which is natural if the curve be a *continuous lemniscate*, or it may be traced as a circuit presenting two coincident salient points. A determinate area requires a determinate mode of making the circuit.

marked. Let the number of times + occurs more than −, or − more than +, be called the *balance* in favour of + or of −. In the line marked in the diagram the balance is 1 in favour of +.

Now let the line revolve round the point O. This balance remains unaltered. For it never alters except by the introduction or dismissal of a pair or pairs, each pair having $\pm$. The losses or gains are at the corners of polygons, or at cusps, or where the revolving line becomes tangent to the circuit.

Again, let any one portion in which we can get from one point to another without crossing the boundary be called a *primary part*. Take any two points in one and the same primary part: the balances of these two points must be the same. Join them: then, taking the line of junction (produced) as that on which the balances are to be made up, it is clear from the sameness of the compartment in which the two points are, that the line of junction is either not crossed at all by the boundary, or else pairs of times by the immediate boundary of the compartment, and it may be, by other parts of the circuit; but always in pairs of times, with $\pm$ in each pair. Consequently, the balances of all points in the same primary part are the same.

Now, *whatever may be the pole*, the manner in which any primary part enters into the area is determined solely by the balance of a line drawn from a pole *in that primary part.* If A be the number of square units in that part, then if the balance be m in favour of +, the part enters as $+mA$; if in favour of −, as $-mA$; while, if there be no balance in favour of either, that primary part contributes nothing to the area, and must be considered as *external to the circuit.*

To show this, take an element dA in any one primary part, and, setting out at the pole of $\frac{1}{2}\int r^2 d\theta$, draw a line to any point of the element dA, and then produce that line in the direction in which it was drawn, past all the boundary. Then, a very little consideration will shew that in the whole sweep of the radius vector by which the circuit is described, dA will enter $+1$ times for every + in the items from which the balance is made, and -1 times for every −. And, by the permanence of the balance, we see that the area must be the same for every pole: that is, that dA must enter in the same way whatever pole may be taken.

In the diagram, the primary parts are all marked so as to indicate the number of times each enters, and the sign of each, on the suppositions that the positive direction of revo-

lution is as indicated in the corner, and that the circuit is made with the arrows. The parts marked 0 are *external;* those marked -2 diminish the area by twice their ordinary area; those marked $+2$ increase the area by twice their ordinary area, and so on. It will easily be seen that (as must be) the index is changed by -1 or $+1$ in passing over a line; by -2, 0, or $+2$, in passing through a double point; by -3, -1, $+1$, or $+3$, in passing through a triple point; and so on. The change in each case is easily determined by the directions of revolution of the parts of the circuit crossed over.

If there exist any point round which the whole circuit can be made positively, then, for that mode of making the circuit (as distinguished from the only other) all the area is positive; and each element enters as often as it is enveloped.

The rectilinear area $\int_a^b y\,dx$ is, in this system, referred to a pole at an infinite distance on the axis of y. The circuit is made (if $b > a$) from $x = a$ up the ordinate to a, along the curve, down the ordinate to b, and then negatively along $b - a$ on the line of abscissæ. The signs of our new convention are then as in the usual interpretation of this formula, for usual cases; and can easily be determined in others. Positive revolution has become positive motion parallel to the axis of x, or one which has a positive component in the direction of x.

University College, London, Oct. 9, 1849.

ON THE POTENTIAL OF A CLOSED GALVANIC CIRCUIT OF ANY FORM.

By William Thomson.

The object of the following note is to point out an extremely interesting application of the principles explained by Professor De Morgan in the preceding paper, which occurred to me in connexion with the determination of the potential of an electro-magnet in terms of the solid angle of a cone.

It has been shewn by Ampère that a *closed galvanic circuit in a re-entering curve of any form* produces the same magnetic action as any infinitely thin sheet of steel, having this curve for its edge, would produce if uniformly and normally mag-

netized. Now the resultant force of a magnet at any point may be expressed, after the manner of Laplace, in terms of the differential coefficients of a "potential function," and therefore the same proposition is true for a closed galvanic circuit.* When this is known to be true, for either a common or an electro-magnet, the following definition may be laid down.

The potential at any point in the neighbourhood of a magnet is the quantity of work necessary to bring a unit north-pole (or the north-pole of an infinitely thin uniformly and longitudinally magnetized unit-bar) from an infinite distance to that point. To determine the potential at any point due to a given closed galvanic circuit, let us imagine a magnetized sheet of steel (the form of the sheet is arbitrary, provided only that its edge coincide with the curve of the galvanic circuit), which according to Ampère produces the same magnetic action, and consequently the same potential, as a given closed circuit, to be divided into infinitely small areas. Then it is easily demonstrated, on the most elementary principles of the theory of magnetism, that the potentials at any point, P, produced by these areas, are proportional to the solid angles which they subtend at P; the true *sign* of the potential of any small area being obtained by considering the solid angle as positive, if the side of the area containing north poles, or negative, if the other side be towards P. Hence the potential of the whole sheet of steel, at any point P, is proportional to the entire solid angle which it subtends at P; and consequently the

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* In other words, the quantity of *work* necessary to bring a magnetic pole from any position in the neighbourhood of a closed galvanic circuit to any other position does not *vary* with the form of the curve along which it is drawn from one point to the other. There is however one remarkable difference between the case of an electro-magnet and that of any given steel magnet. In the case of an electro-magnet, although the quantity of work does not *vary* with the path, yet it has determinately different values according as the path lies on one side, or on another of any part of the galvanic wire circuit, or according to the convolutions round any part of the wire which it may be arbitrarily chosen to make. Hence arises the multiplicity of values of the potential at any point in the neighbourhood of an electro-magnet noticed below. Yet for any one form of a magnetized sheet of steel of the kind described in the text, agreeing, in the action which it produces on all points not in its own substance, with the electro-magnet, the potential is perfectly determinate without a multiplicity of values; and the difference in the two cases is accounted for when we consider that the magnetic potentials at any two points infinitely near one another, on two sides of the sheet of steel, differ by $4\pi\gamma$, where γ is a constant such that $\gamma\omega$ is the magnetic moment of any infinitely small area ω of the sheet. The agreement in the magnetic circumstances in the two cases fails for all points in the substance of the magnetized steel.

potential of a closed galvanic circuit, at any point P, is equal to a constant (which may be taken as a measure of the strength of the galvanism, or as it is often termed, the "quantity" of the current) multiplied into the solid angle of the cone described by a straight line always passing through P, and carried round the circuit. In all cases, except those in which the galvanic circuit is contained in one plane, there will be positions of P for which this cone will be "autotomic"; and in many cases, especially the most common practical case of an electro-magnet, in which the circuit consists of double or multiple concentric helices, with their ends connected, or of a single wire wrapped in a complex manner round a body of some irregular shape, so as to constitute most complicated curves of double curvature, there will be no position of the point P for which the cone is not excessively autotomic. The solid angle of such a cone, or the area enclosed by its intersection with a spherical surface of unit radius, having for centre its vertex, may be determined in a manner precisely similar to that which has been explained by Professor De Morgan for plane self-cutting curves, without any ambiguity as to the *circuit* by which the curve, when self-cutting,* is to be described, since the actual galvanic current is in a determinate circuit, and its projection, by the conical surface, on the surface of the sphere is to be described by the projection of a point moving along the electric conductor, either in the same direction as the current, or in the opposite, according to the convention we please to make. There is however a source of ambiguity which really affects the evaluation of the solid angle of a cone, or of the area of any given circuit described in a determinate manner on a spherical surface, and gives rise to a multiplicity of solutions of the problem, arising from the circumstance that of all the "primary parts" (only two in number if the circuit be not self-cutting) into which the spherical surface is divided by the curve, there is no reason for choosing one more than another as a zero space (or a space corresponding to the space exterior to a closed circuit in a plane).†

When the value of the area, according to any one of these solutions, has been obtained, all the others may be deduced, by adding to it or subtracting from it any number of times

* See note on the word 'circuit' (p. 140) in the preceding paper.

† Thus, if the given curve be a circle of the sphere, described in a given direction, and if θ denote the angular radius measured from that pole O, which would be *north* if the direction of describing the circle were from

the area of the whole spherical surface. Hence the most general expression for the solid angle of a cone described in a determinate manner, is

$$\sigma = \sigma_1 + 4i\pi,$$

where σ_1 denotes any one value and i any positive or negative integer. If too great a positive or too small a negative value be given to i, all the "primary spaces" of the spherical surface will be positive or all will be negative; and therefore if we wish to obtain only those solutions according to which some portion of the spherical surface is considered as *zero* or *external to the circuit*, a limited number only (not exceeding the number of primary parts into which the spherical surface is divided by the circuit) of values for i are to be admitted. The physical problem, however, requires no limitation to the range of values that may be given to i: for, if we take any two paths to the point P from an infinite distance, such that the space between them is *once* crossed by the galvanic circuit, the potential at P will differ by $4\pi\gamma$ according as it is estimated by one path or by the other; and therefore, by taking (for the sake of simplicity in the conception) different paths to the point P which go round a certain portion of the galvanic circuit once, twice, three times, four times, &c. in one direction, and again different paths which go round the same portion of the wire once, twice, three times, four times, &c. in the contrary direction, we obtain, according to the definition, an infinite number of values of the potential at the point P, which are successively expressed by the formula

$$v = v_1 + 4i\pi\gamma,$$

when we give i the values 1, 2, 3, 4, &c., and again the values -1, -2, -3, -4, &c.; v_1 being the potential estimated by a path, which makes none of the convolutions of the kind described with reference to the others.

Hence we see that, to find the general expression for the potential at a point in the neighbourhood of an electro-magnet, we may first choose some determinate path from an infinite distance to the point P, and investigate the value of

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west to east; the area of the circuit is $2\pi\,(1 - \cos\theta)$ if the space on the other side of the circle from O be considered as the zero space, but it would be $-2\pi\,(1 + \cos\theta)$ if the space in which O is situated were taken as zero, or external to the circuit. In general, the area of a circuit not self-cutting, on a spherical surface, will be either, one of the two parts into which the spherical surface is divided, with the sign $+$, or the other part, with the sign $-$.

the potential for it, which may be used as the value of v_1 in the preceding expression. If an infinite straight line in any direction, terminated at the point P, be the path chosen, the determinate potential will be found by considering the primary portion of the spherical surface described from P as centre, which is cut by this line, as the portion external to the circuit. Hence, if we mark this primary portion with a zero, the number with which any other primary part is to be marked, according to Professor De Morgan's rule, will be got by drawing a line to any point within it, from any point O, in the external primary part, and counting the number of times it is cut by the curve; every time it is cut from right to left (with reference to a person walking from O, along it, on the convex surface of the sphere) being counted as $+1$, and every time it is cut in the other direction, as -1, and the algebraical sum taken. When the number for each primary part has been thus determined, the sum of the areas, each multiplied by its number, (positive or negative, as the case may be) of the different primary parts, will be the required area of the circuit; and the potential at the centre of the sphere will be obtained by multiplying this by γ, the strength of the galvanic current. The absolute sign of the potential thus determined may be readily shewn to be correct, if we agree to consider the potential due to terrestrial magnetism as on the whole positive for positions north, and negative for positions south of the magnetic equator; since, as is well known, currents round the earth, proceeding on the whole from east to west, would produce phenomena similar to the actual phenomena of terrestrial magnetism.

As an example, let us consider a conducting circuit which consists of twelve complete spires of a helix, and a line along the axis with two perpendicular portions connecting its extremities with those of the helix. The accompanying diagrams represent the projections, by radii, of the circuit, on a spherical surface in two different positions, viewed in each case from the interior of the sphere.

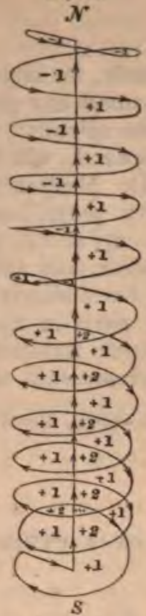
In the case illustrated by fig. (1), the centre of the sphere is nearly in a line with the axis of the helix, on the side towards the north pole* of the helix, and distant from it by about half the length of the axis. In the case illustrated by

* The ends of the helix which would be repelled from the north and from the south respectively by the earth's magnetic action are, in the ordinary vague use of the term "pole," called the north and south poles of the electro-magnet.

fig. (2), the centre of the sphere is in a perpendicular through a point of the axis, distant by about one-fourth of its length from the north pole of the helix, and is at about the same

Fig. 1.

Fig. 2.



distance from the nearest part of the helix, as in the case of fig. (1); and the curve on the spherical surface is shewn in the diagram, according to Mercator's projection with the great circle containing the axis of the helix as equator.* In each diagram the inner side of the spherical surface is shown.

The radii of the spheres being supposed to be equal in the two cases, if we denote their common value by r, and if A_1 and A_2 be the areas of the spherical curves represented in the diagrams, the zero or external portions on the

* The diagram was actually drawn by tracing the shadow of a helix of twelve spires, $\frac{3}{4}$ in. in diameter and 4 in. in length, upon a cylindrical surface, produced by a luminous point at its centre; the axis of the helix being held in a plane through the luminous point at right angles to the axis of the cylindrical surface. On account of the narrowness of the band occupied by the diagram, the cylindrical surface very nearly coincided with the spherical surface, which in strictness ought to have received the shadow. After the shadow was thus traced, the cylindrical surface was unbent into a plane.

spherical surfaces being taken as those which become infinite in the plane diagrams, the values of the potential at the centre of the sphere will be

$$\gamma \frac{A_1}{r^2}, \text{ and } \gamma \frac{A_2}{r^2},$$

respectively, for any paths from an infinite distance which do not lie round any portion of the galvanic wire, nor between any of the spires.

The area A_1 will be determined (in accordance with Professor De Morgan's rule*) by finding the areas of the "primary parts," marked successively with the numbers 1, 2, ... up to 12, multiplying each area by the corresponding number, and taking the sum of the products. The area A_2 will be similarly determined by finding the areas of the primary parts in fig. (2), multiplying each by the positive or negative number with which it is marked, and taking the algebraic sum of the products.

Glasgow College, March 25, 1850.

NOTE ON A FAMILY OF CURVES OF THE FOURTH ORDER.

By Arthur Cayley.

The following theorem, in a slightly different and somewhat less general form, is demonstrated in Mr. Hearn's "Researches on Curves of the Second Order," &c., *Lond.* 1846: "The locus of the pole of a line, $u + v + w = 0$, with respect to the conics passing through the angles of the triangle ($u = 0$, $v = 0$, $w = 0$), and touching a fixed line $\alpha u + \beta v + \gamma w = 0$, is the curve of the fourth order,

$$\sqrt{\{\alpha u (v + w - u)\}} + \sqrt{\{\beta v (w + u - v)\}} + \sqrt{\{\gamma w (u + v - w)\}} = 0 \text{;"}$$

the difference in fact being, that with Mr. Hearn the inde-

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* In fig. (1), all the arrow-heads which are necessary for rendering determinate the "balances" for the primary parts are given; and the numbers expressing the balances are marked for the first six primary parts, commencing with the outermost. In fig. (2), all the arrow-heads which are necessary to make the diagram represent determinately a closed circuit are indicated, except in a few places where the spaces are too confined for admitting of this being done in a clear manner; and the "balances" of all the primary parts are marked with numbers, except in the instance of a very small *triple* primary part, which is marked with three dots (...) instead of + 3.

terminate line $u+v+w=0$ is replaced by the line ∞, so that the poles in question become the centres of the conics.

Previous to discussing the curve of the fourth order, it will be convenient to notice a property of curves of the fourth order with three double points. Such curves contain eleven arbitrary constants: or if we consider the double points as given, then five arbitrary constants. From each double point may be drawn two tangents to the curve; any five of the points of contact of these tangents determine the curve, and consequently determine the sixth point of contact. The nature of this relation will be subsequently explained; at present it may be remarked that it is such that, if three of the points of contact (each one of such points of contact corresponding to a different double point) lie in a straight line, the remaining three points of contact also lie in a straight line. A curve of the fourth order having three given double points and besides such that the points of contact of the tangents from the double points lie three and three in two straight lines, contains therefore four arbitrary constants. Now it is easily seen that the curve in question has three double points, viz. the points given by the equations

$$(u=0,\ v-w=0),\ (v=0,\ w-u=0),\ (w=0,\ u-v=0),$$

points which may be geometrically defined as the projections from the angles of the triangle $(u=0,\ v=0,\ w=0)$ upon the opposite sides, of the point $(u=v=w)$ which is the harmonic with respect to the triangle of the given line $u+v+w=0$. Moreover, the lines $u=0$, $v=0$, $w=0$ (being lines which, as we have seen, pass through the double points) touch the curve in three points lying in a line, viz. the given line $\alpha u+\beta v+\gamma w=0$. Hence the curve in question is a curve with three double points, such that the points of contact of the tangents from the double points lie three and three in two straight lines. Considering the double points as given, the functions u, v, w contain two arbitrary ratios, and the ratios of the quantities α, β, γ being arbitrary, the equation of the curve contains four arbitrary constants, or it represents the general curve of the class to which it has been stated to belong.

As to the investigation of the above-mentioned theorem with respect to curves of the fourth order with three double points, the general form of the equation of such a curve is

$$\frac{a}{x^2}+\frac{b}{y^2}+\frac{c}{z^2}+\frac{2f}{yz}+\frac{2g}{zx}+\frac{2h}{xy}=0,$$

where the double points are the angles of the triangle $(x=0,\ y=0,\ z=0)$. It may be remarked in passing, that the six tangents at the double point touch the conic

$$ax^2 + by^2 + cz^2 - 2fyz - 2gxz - 2hxy = 0.$$

To determine the tangents through $(y=0,\ z=0)$, we have only to write the equation to the curve under the form

$$\left(\frac{a}{x} + \frac{h}{y} + \frac{g}{z}\right)^2 + \frac{C}{y^2} + \frac{B}{z^2} - \frac{2F}{yz} = 0:$$

the points of contact are given by the system

$$\frac{a}{x} + \frac{h}{y} + \frac{g}{z} = 0,$$

$$\frac{C}{y^2} + \frac{B}{z^2} - \frac{2F}{yz} = 0;$$

the latter equation (which evidently belongs to a pair of lines) determining the tangents. The former equation is that of a conic passing through the angles of the triangle $x=0,\ y=0,\ z=0$: since the tangents pass through the point $(y=0,\ z=0,)$ they evidently each intersect the conic in one other point only. The equation of the tangents shews that these lines are the tangents through the point $y=0,\ z=0$ to the conic, whose equation is

$$aA^2x^2 + bB^2y^2 + cC^2z^2 + 2fBCyz + 2gCAzx + 2hABxy = 0.$$

To complete the construction of the points of contact it may be remarked, that the equations which determine the coordinates of these points may be presented under the form

$$Ax = \{A \qquad\qquad\qquad\} x,$$

$$By = \left\{H - g\sqrt{\left(\frac{-k}{a}\right)}\right\} x,$$

$$Cz = \left\{G + h\sqrt{\left(\frac{-k}{a}\right)}\right\} x:$$

whence also

$$aAx + hBy + gCz = Kx,$$

$$hAx + bBy + fCz = G\sqrt{\left(\frac{-k}{a}\right)}\, x,$$

$$gAx + fBy + cCz = -H\sqrt{\left(\frac{-k}{a}\right)}\, x:$$

or writing for shortness $\xi,\ \eta,\ \zeta$ instead of the linear functions

forming the first sides of these equations respectively, we have

$$\frac{\xi}{\sqrt{(-aK)}} = \frac{\eta}{G} = -\frac{\zeta}{H};$$

from which it follows at once that the equation to the line forming the two points of contact is

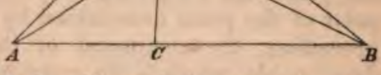
$$\frac{\eta}{G} + \frac{\zeta}{H} = 0.$$

And this may again be considered as the line joining the points $(\xi = 0,\ \frac{\xi}{F} + \frac{\eta}{G} + \frac{\zeta}{H} = 0)$ and $(\eta = 0,\ \zeta = 0)$.

Now $\xi = 0$, $\eta = 0$, $\zeta = 0$, are the polars of the double points (or angles of the triangle $x = 0$, $y = 0$, $z = 0$) with respect to the last-mentioned conic. The equation of the line joining the point $(y = 0,\ z = 0)$ with the point $(\eta = 0,\ \zeta = 0)$, is easily seen to be $GBy - CHz = 0$; from which it follows, that the lines forming each double point with the intersection of the polars of the other two double points meet in a point, the coordinates of which are given by

$$x : y : z = \frac{1}{AF} : \frac{1}{BG} : \frac{1}{CH},$$

and the polar of this point is the line $\frac{\xi}{F} + \frac{\eta}{G} + \frac{\zeta}{H} = 0$. The construction of this line is thus determined; and we have already seen how this leads to the construction of the lines joining the points of contact of the tangents from the double points. In fact, in the figure, if $\alpha\beta\gamma$ be the triangle whose sides are $\xi = 0$, $\eta = 0$, $\zeta = 0$, and A, B, C the points of intersection of the sides of this triangle with the line



$$\frac{\xi}{F} + \frac{\eta}{G} + \frac{\zeta}{H} = 0,$$

the lines in question are $A\alpha$, $B\beta$, $C\gamma$. Moreover, the points of contact upon $A\alpha$ are harmonics with respect to A, α, and in like manner the points of contact on $B\beta$, $C\gamma$ are respectively harmonics with respect to B, β and C, γ. This proves

that if three of the points of contact are in the same straight line, the remaining three are also in the same straight line; in fact, we may consider three of the points of contact as given by

$$\xi : \eta : \zeta = \sqrt{(-aK)} : \quad G \quad : -H,$$
$$\xi : \eta : \zeta = \quad -F \quad : \sqrt{(-bK)} : H,$$
$$\xi : \eta : \zeta = \quad F \quad : \quad -G \quad : \sqrt{(-cK)};$$

and the condition that these may be in the same line is

$$\sqrt{(-a)}\sqrt{(-b)}\sqrt{(-c)} + GH\sqrt{(-a)} + HF\sqrt{(-b)} + FG\sqrt{(-c)} = 0,$$

which remains unaltered when the signs of all the roots are changed. The equation just obtained may be considered as the condition which must exist between the coefficients of

$$\frac{a}{x^2} + \frac{b}{y^2} + \frac{c}{z^2} + \frac{2f}{yz} + \frac{2g}{zx} + \frac{2h}{xy} = 0,$$

in order that this curve may be transformable into the form

$$\sqrt{\{\alpha u(v+w-u)\}} + \sqrt{\{\beta v(w+u-v)\}} + \sqrt{\{\gamma w(u+v-w)\}} = 0.$$

ON THE DEVELOPABLE DERIVED FROM AN EQUATION OF THE FIFTH ORDER.

By Arthur Cayley.

Möbius, in his "Barycentrische Calcul," has considered, or rather suggested for consideration, the family of curves of double curvature given by equations such as $x : y : z : w = A : B : C : D$, where A, B, C, D are rational and integral functions of an indeterminate quantity t. Observing that the plane $Ax + By + Cz + Dw = 0$ may be considered as the polar of the point determined by the system of equations last preceding, the reciprocal of the curve of Möbius is the developable, which is the envelope of a plane the coefficients in the equation of which are rational and integral functions of an indeterminate quantity t, or what is equivalent, homogeneous functions of two variables ξ, η. Such an equation may be represented by $U = a\xi^n + n\xi^{n-1}\eta + \ldots. = 0$, (where a, b, &c. are linear functions of the coordinates); and we are thus led to the developables noticed, I believe for the first time, in my "Note sur les Hyperdeterminants," *Crelle*, tom. XXXIV. p. 148. I there remarked, that not only the equation of the developable was to be obtained by elimi-

ting ξ, η from the first derived equations of $U = 0$; but that the second derived equations conducted in like manner to the edge of regression, and the third derived equations to the cusps or stationary points of the edge of regression. It followed that the order of the surface was $2(n-1)$, that of the edge of regression $3(n-2)$, and the number of stationary points $4(n-3)$. These values lead at once, as Mr. Salmon pointed out to me, to the table,

$$
\begin{aligned}
m &= 3(n-2),\\
n &= n,\\
r &= 2(n-1),\\
\alpha &= 0,\\
\beta &= 4(n-3),\\
g &= \tfrac{1}{2}(n-1)(n-2),\\
h &= \tfrac{1}{2}(9n^2 - 53n + 80),\\
x &= 2(n-2)(n-3),\\
y &= 2(n-1)(n-3),
\end{aligned}
$$

where the letters in the first column have the same signification as in my memoir in Liouville, translated in the last number of the *Journal*. The order of the nodal line is of course $2(n-2)(n-3)$; Mr. Salmon has ascertained that there are upon this line $6(n-3)(n-4)$ stationary points and $\frac{4}{3}(n-3)(n-4)(n-5)$ real double points, (the stationary points lying on the edge of regression, and with the stationary points of the edge of regression forming the system of intersections of the nodal line and edge of regression, and the real double points being triple points upon the surface). Also, that the number of apparent double points of the nodal line is $(n-3)(2n^3 - 18n^2 + 57n - 65)$.

The case of U a function of the second order gives rise to the cone $ac - b^2 = 0$. When U is a function of the third order, we have the developable

$$4(ac - b^2)(bd - c^2) - (ad - bc)^2 = 0,$$

which is the general developable of the fourth order having for its edge of regression the curve of the third order,

$$ac - b^2 = 0, \quad bd - c^2 = 0, \quad ad - bc = 0,$$

which is likewise the most general curve of this order: there are of course in this case no stationary points on the edge of regression. In the case where U is of the fourth order we have the developable of the sixth order,

$$(ae - 4bd + 3c^2)^3 - 27(ace + 2bcd - ad^2 - b^2e - c^3)^2 = 0;$$

having for its edge of regression the curve of the sixth order,

$$ae - 4bd + 3c^2 = 0, \quad ace + 2bcd - ad^2 - b^2e - c^3 = 0,$$

with four stationary points determined by the equations

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e}.$$

The form exhibiting the nodal line of the surface has been given in the *Journal* by Mr. Salmon. I do not notice it here, but pass on to the principal subject of the present paper, which is to exhibit the edge of regression and the stationary points of this edge of regression for the developable obtained from the equation of the fifth order,

$$U = a\xi^5 + 5b\xi^4\eta + 10c\xi^3\eta^2 + 10d\xi^2\eta^3 + 5e\xi\eta^4 + f\eta^5 = 0;$$

viz. that represented by the equation

$$\begin{aligned}
\square = 0 = {} & a^4f^4 + 160a^3ce^2f^2 + 16a^2b^2df^3 + 360a^3d^2ef^2 + 360a^2bc^2f^3 \\
& \qquad + 2640a^2c^2d^2f^2 \\
& + 256a^3e^5 + 256b^5f^3 + 320a^2bce^3f + 320ab^3def^2 \\
& \qquad + 4080a^2bd^2e^2f + 4080ab^2c^2ef^2 \\
& + 5760a^2cd^2e^3 + 5760b^3c^2df^2 + 3456a^2d^5f + 3456ac^5f^2 \\
& \qquad + 4480a^2c^2de^2f + 4480ab^2cd^2f^2 \\
& + 7200ab^2ce^4 + 7200b^4de^2f + 960ab^2d^3ef + 960abc^3e^2f \\
& \qquad + 28480abc^2d^2ef \\
& + 7200abcd^3e^2 + 7200b^2c^3def + 5120ac^3d^3f + 6400ac^4e^3 + 6400b^3d^4f \\
& + 9000b^3cde^3 + 2000b^2c^2d^2e^2, \\
& - 3375b^4e^4 - 20a^3bef^3 - 120a^3cdf^3 - 640a^3de^3f - 640ab^3cf^3 \\
& \qquad - 10a^2b^2e^2f^2 - 1640a^2bcdef^2 \\
& - 1440a^2bd^3f^2 - 1440a^2c^3ef^2 - 1920a^2bde^4 - 1920b^4cef^2 \\
& \qquad - 2560a^2c^2e^4 - 2560b^4d^2f^2 \\
& - 10080a^2cd^3ef^2 - 10080abc^3df^2 - 2160a^2d^4e^2 - 2160b^2c^4f^2 \\
& \qquad - 180ab^3e^3f - 14920ab^2cde^2f \\
& - 600ab^2d^2e^3 - 600b^3c^2e^2f - 3200ac^3d^2e^2 - 3200b^2c^2d^3f \\
& \qquad - 11520abcd^4f - 11520ab^4def \\
& - 16000abc^2de^3 - 16000b^3cd^2ef - 4000b^3d^3e^2 - 4000b^2c^3e^3
\end{aligned}$$

a result for which I am indebted to Mr. Salmon.

To effect the reduction of this expression, consider in the first place the equations which determine the stationary

points of the edge of regression. Writing instead of $\xi : \eta$ the single letter t, these equations are

$$at^2 + 2bt + c = 0,$$
$$bt^2 + 2ct + d = 0,$$
$$ct^2 + 2dt + e = 0,$$
$$dt^2 + 2et + f = 0:$$

write for shortness

$$A = 2\,(bf - 4ce + 3d^2)$$
$$B = \ \ af \ \ - 3be + 2cd$$
$$C = 2\,(ae - 4bd + 3c^2)$$

and let α, 3β, 3γ, δ represent the terms of

$$\left\|\begin{matrix} a, & b, & c, & d \\ b, & c, & d, & e \\ c, & d, & e, & f \end{matrix}\right\|$$

viz.

$$\alpha = bdf - be^2 + 2cde - c^2f - d^3,$$
$$3\beta = adf - ae^2 - bcf + bde + c^2e - cd^2,$$
$$3\gamma = acf - ade - b^2f + bd^2 + bce - c^2d,$$
$$\delta = ace - ad^2 - b^2e + 2bcd - c^3.$$

It is obvious at first sight that the result of the elimination of t from the four quadratic equations is the system (equivalent of course to three equations),

$$\alpha = 0, \quad \beta = 0, \quad \gamma = 0, \quad \delta = 0.$$

The system in question may however be represented under the simpler form (which shews at once that the number of stationary points is as it ought to be eight),

$$A = 0, \quad B = 0, \quad C = 0,$$

as appears from the identical equations,

$$\begin{aligned} &(2ct + 3d)\,(bt^2 + 2ct + d), \\ &- (2bt + 4c)\,(ct^2 + 2dt + e), \\ &+ b\,(dt^2 + 2et + f) = \tfrac{1}{2}A\,; \end{aligned}$$

$$\begin{aligned} &(2ct + 3d)\,(at^2 + 2bt + c), \\ &\qquad - c\,(bt^2 + 2ct + d), \\ &- (2at + 3b)\,(ct^2 + 2dt + e), \\ &\qquad + a\,(dt^2 + 2et + f) = B\,; \end{aligned}$$

$$\begin{aligned} &(2bt + 3c)\,(at^2 + 2bt + c), \\ &- (2at + 4b)\,(bt^2 + 2ct + d), \\ &+ a\,(ct^2 + 2dt + e) = \tfrac{1}{2}C, \end{aligned}$$

(formulæ the first and third of which are readily deduced from an equation given in the Note on Hyperdeterminants above quoted). The connexion between the quantities A, B, C and $\alpha, \beta, \gamma, \delta$, is given by

$$\begin{aligned} Aa - 2Bb + Cc &= -6\delta, \\ Ab - 2Bc + Cd &= -6\gamma, \\ Ac - 2Bd + Ce &= -6\beta, \\ Ad - 2Be + Cf &= -6\alpha. \end{aligned}$$

The theory of the stationary points being thus obtained, the next question is that of finding the equations of the edge of regression. We have for this to eliminate t from the three cubic equations,

$$\begin{aligned} at^3 + 3bt^2 + 3ct + d &= 0, \\ bt^3 + 3ct^2 + 3dt + e &= 0, \\ ct^3 + 3dt^2 + 3et + f &= 0: \end{aligned}$$

treating the quantities t^3, t^2, t^1, t^0 as if they were independent, we at once obtain

$$\beta t + \alpha = 0, \quad \delta t + \gamma = 0, \quad \gamma t^2 - \alpha = 0, \quad \delta t^2 - \beta = 0;$$

or as this system may be more conveniently written,

$$\beta t + \alpha = 0, \quad \gamma t + \beta = 0, \quad \delta t + \gamma = 0.$$

But the most simple forms are obtained from the identical equations,

$$\begin{aligned} & ft\,(at^3 + 3bt^2 + 3ct + d), \\ & -(3et + f)\,(bt^3 + 3ct^2 + 3dt + e), \\ & +(2dt + e)\,(ct^3 + 3dt^2 + 3et + f) = t^3\,(Bt + A); \\ & (bt + c)\,(at^3 + 3bt^2 + 3ct + d), \\ & -(at + 3b)\,(bt^3 + 3ct^2 + 3dt + e), \\ & +\quad a \quad (ct^3 + 3dt^2 + 3et + f) = Ct + B; \end{aligned}$$

equations which, combined with those which precede, give the complete system

$$\beta t + \alpha = 0, \quad \gamma t + \beta = 0, \quad \delta t + \gamma = 0, \quad Bt + A = 0, \quad Ct + B = 0:$$

or the equations of the edge of regression are given by the system (equivalent of course to two equations),

$$\left\| \begin{matrix} \alpha, & \beta, & \gamma, & A, & B \\ \beta, & \gamma, & \delta, & B, & C \end{matrix} \right\| = 0.$$

The simplest mode of verifying *a posteriori* that the edge of regression is only of the ninth order, appears to be to con-

sider this curve as the common intersection of the three surfaces of the seventh order:

$$A^3a - 3A^2Bb + 3AB^2c - B^3d = 0,$$
$$A^3b - 3A^2Bc + 3AB^2d - B^3e = 0,$$
$$A^3c - 3A^2Bd + 3AB^2e - B^3f = 0,$$

(which are at once obtained by combining the equation $Bt + A = 0$ with the cubic equations in t). It is obvious from a preceding equation that if the equations first given are multiplied by fA, $-3eA + fB$, $2dA - eB$, and added, an identical result is obtained. This shews that the curve of the forty-ninth order, the intersection of the first two surfaces, is made up of the curve in question, the curve of the fourth order $A = 0$, $B = 0$ (which reckons for thirty-six, as being a triple line on each surface), and the curve which is common to the two surfaces of the seventh order and the surface $2dA - eB = 0$. The equations of this last curve may be written,

$$e(af - 3be + 2cd) - 4d(bf - 4ce + 3d^2) = 0,$$
$$e^3a - 6e^2db + 12ed^2c - 8d^4 = 0,$$
$$e^3b - 6e^2dc + 4ed^3 = 0.$$

Or, observing that these equations are

$$f(ae - 4bd) - 3(be^2 - 6ced + 4d^3) = 0,$$
$$e^2(ae - 4bd) - 2d(be^2 - 6ced + 4d^3) = 0,$$
$$e(be^2 - 6ced + 4d^3) = 0;$$

the last-mentioned curve is the intersection of

$$ae - 4bd = 0,$$
$$be^2 - 6ced + 4d^3 = 0,$$

where the second surface contains the double line $e = 0$, $d = 0$, which is also a single line upon the first surface. Omitting this extraneous line, the intersection is of the *fourth* order; and we may remark that, in passing, it is determined (exclusively of the double line) as the intersection of the three surfaces

$$ae - 4bd = 0,$$
$$be^2 - 6ced + 4d^3 = 0,$$
$$a^2d - 6abc + 4b^3 = 0,$$

being in fact of the species IV. 4. of Mr. Salmon's paper 'On the Classification of Curves of Double Curvature.' But returning to the question in hand, Since $49 = 9 + 4$ -

we see that the curve common to the three surfaces of the seventh order, or the edge of regression, is, as it ought to be, of the ninth order. It only remains to express the equation of the developable surface in terms of the functions A, B, C, α, β, γ, δ, which determine the stationary points and edge of regression; I have satisfied myself that the required formula is

$$\square = (AC - B^2)^2 - 1152\{A(\beta\delta - \gamma^2) + B(\gamma\beta - \alpha\delta) + C(\alpha\gamma - \beta^2)\} = 0,$$

where the quantities α, β, γ, δ may be replaced by their values in A, B, C; and it will be noticed that when this is done, the terms of $\square$ are each of them at least of the third order in the last-mentioned functions.

I propose to term the family of developables treated of in this paper 'planar developables.' In general, the coefficients of the generating plane of a developable being algebraical functions of a variable parameter t, the equation rationalized with respect to the parameter belongs to a system of n different planes; the developable which is the envelope of such a system may be termed a 'multiplanar developable,' and in the particular case of n being equal to unity, we have a planar developable. It would be very desirable to have some means of ascertaining from the equation of a developable what the degree of its 'planarity' is.

P.S.—At the time of writing the preceding paper I was under the impression that the only surface of the fourth order through the edge of regression was that given by the equation $AC - B^2 = 0$; but Mr. Salmon has since made known to me an entirely new form of the function $\square$, the component functions of which, equated to zero, give six different surfaces of the fourth order, each of them passing through the edge of regression. The form in question is

$$3\square = LL' + 64MM' - 64NN',$$

where

$$\begin{aligned} L &= a^2f^2 + 225b^2e^2 - 32ace^2 - 32b^2df + 480bd^3 + 480c^3e - 34abef \\ &\qquad + 76acdf - 12bc^2f - 12ad^2e - 320c^2d^2 - 820bcde, \\ L' &= 3a^2f^2 - 45b^2e^2 + 64ace^2 + 64b^2df - 22abef - 12acdf \\ &\qquad - 36bc^2f - 36ad^2e + 20bcde, \\ M &= 10bcd^2 - 18ad^3 - 15bc^2e + 32acde + 6b^2cf - 9ac^2f \\ &\qquad + 2abdf + a^2ef - 9abe^2, \\ M' &= 10c^2de - 18c^3f - 15bd^2e + 32bcdf + 6ade^2 - 9b^2ef \\ &\qquad + 2acef + abf^2 - 9b^2ef, \end{aligned}$$

$$N = 10b^2d^2 - 15b^2ce - 12acd^2 + 18ac^2e + abde - 2a^2e^2 + 6b^3f - 9abcf + 3a^2df,$$

$$N' = 10c^2e^2 - 15bde^2 - 12c^2df + 18bd^2f + bcef - 2b^2f^2 + 6ae^3 - 9adef + 3acf^2,$$

where it should be noticed that

$$L + 3L' = -10(AC - B^2).$$

The expressions of L, L', M, M', N, N' as linear functions of A, B, C (also due to Mr. Salmon) are

$$L = (11ae + 28bd - 39c^2)A + (af - 75be + 74cd)B + (11bf + 28ce - 39d^2)C,$$

$$L' = (-7ae + 4bd + 3c^2)A + (3af + 15be - 18cd)B + (-7bf + 4ce + 3d^2)C,$$

$$M = 3(bc - ad)A + 3(ae - c^2)B + (cd - af)C,$$

$$M' = (cd - af)A + 3(bf - d^2)B + 3(de - cf)C,$$

$$N = 3(b^2 - ac)A + 3(ad - bc)B + (bd - ae)C,$$

$$N' = (ce - bf)A + 3(cf - de)B + 3(e^2 - df)C.$$

I propose resuming the subject of these forms, and the general theory, in a subsequent paper.

[NOTE BY MR. SALMON.]

THE following remarks were suggested by a perusal of the preceding paper with which I was favoured by Mr. Cayley.

In the preceding paper Mr. Cayley commenced by determining the conditions that the equation of the fifth degree should have four equal roots; by the help of these he formed the condition that it should have three, and finally that it should have two equal roots. So in like manner it is evident that the condition that an equation of any higher degree should have two equal roots must be satisfied *à fortiori*, if it have any greater number of equal roots. If we have found the conditions that the equation should have k equal roots ($A = 0$, $B = 0$, $C = 0$, &c.), then the condition that it should have any smaller number of equal roots must be of the form

$$aA + bB + cC + \&c. = 0.$$

It would seem then, that if we were required to find the conditions that an equation of any degree should have

any number of equal roots, it might be well to start with the conditions (easily obtained) that all the roots of the equation should be equal, and to endeavour to ascend from these to the conditions that all but one, all but two, &c. should be equal; and so on, until we had arrived at the required conditions. I write the general equation

$$a_n t^n + n.a_{n-1}t^{n-1} + \frac{n.(n-1)}{1.2}a_{n-2}t^{n-2} + \frac{n.(n-1)(n-2)}{1.2.3}a_{n-3}t^{n-3} + \&c. = 0.$$

This equation will have $n-k$ equal roots if the equation can be identified with

$$(x+a)^{n-k}(b_k x^k + kb_{k-1}x^{k-1} + \&c.).$$

I form the general term of the latter equation and compare it with that of the former, and I find

$$a_p = a^{n-k-p}(Ap^k + Bp^{k-1} + \&c.),$$

where A, B, &c. are quantities independent of p; consequently any relation between the coefficients which is satisfied when the equation has $n-k$ equal roots, will be satisfied identically when we make for each coefficient the above substitution. Let us for example enquire when it is possible to express such a relation by an equation of the second degree in the coefficients. Such a relation is the sum of a number of terms

$$\Sigma La_p a_q = 0.$$

If we have $p+q$ the same for every term, after the substitution every term will be multiplied by the same power of a; and introducing this relation, it is easy to see that the equation will be identically satisfied if we have

$$\Sigma L = 0, \quad \Sigma Lpq = 0, \quad \Sigma Lp^2q^2 = 0, \ \&c., \ldots \ \Sigma Lp^kq^k = 0.$$

In order then that it may be possible to determine L, &c. so as to satisfy these equations, the equation $\Sigma La_p a_q$ must have at least $k+2$ terms; and the condition that the equation should have $n-k$ equal roots cannot be expressed by an equation of the second degree, when $\frac{1}{2}n+1$ is less than $k+2$, for no number (of the form either $2q$ or $2q+1$) can be formed in more than $q+1$ different ways as the sum of two smaller numbers.

Thus, for example, the condition that all the roots should be equal is simply $L + L_1 = 0$, or is of the form

$$a_p a_q - a_r a_s = 0, \quad \text{where } p+q = r+s.$$

The conditions that $n-1$ roots should be equal are found from the equations

$$L + L' + L'' = 0, \quad pqL + rsL_1 + tuL'' = 0,$$

and must be of the form

$$(rs - tu)\,a_p a_q + (tu - pq)\,a_r a_s + (pq - rs)\,a_t a_u = 0.$$

To this class belong Mr. Cayley's three equations,

$$a_4 a_0 - 4a_3 a_1 + 3a_2 a_2 = 0 \ldots\ldots\ldots\ldots (A),$$

$$a_5 a_0 - 3a_4 a_1 + 2a_3 a_2 = 0 \ldots\ldots\ldots\ldots (B),$$

$$a_5 a_1 - 4a_4 a_2 + 3a_3 a_3 = 0 \ldots\ldots\ldots\ldots (C).$$

And in general the solution of a system of linear equations of the form

$$\begin{aligned} L + \ \ M + \ \ N + \&c. &= 0, \\ aL + bM + cN + \&c. &= 0, \\ a^2L + b^2M + c^2N + \&c. &= 0, \\ \&c. \end{aligned}$$

is $L = \dfrac{1}{(a-b)(a-c)(a-d), \&c.}$, $M = \dfrac{1}{(b-a)(b-c)(b-d)}$;

or else we may write it

$$L = (b-c)(b-d)(d-c), \&c.,$$

or equal to the product of all the differences which do not contain a, &c.

We can thus deduce the following equation as one form of the condition that an equation should have $n-k$ equal roots. (I use the factorial notation $[n]^k$ for a product of k factors $n.(n-1)\ldots.(n-k+1)$),

$$\begin{aligned} &[n-k-2]^k a_n a_0 - (k+1)(n-2)[n-k-3]^{k-1} a_{n-1} a_1 \\ &\quad + \frac{(k+1)k}{1.2}(n-4)[n-1]_1 [n-k-4]^{k-2} a_{n-2} a_2, \\ &- \frac{(k+1)k.(k-1)}{1.2.3}(n-6)[n-1]^2 [n-k-5]^{k-3} a_{n-3} a_3 + \&c. = 0. \end{aligned}$$

The coefficients in the preceding equation will not alter if all the subindices be increased by any constant quantity. For we have proved that the coefficients depend only on the differences

$pq - rs$ but $(p+\phi)(q+\phi) - (r+\phi)(s+\phi) = pq - rs$,

since $\qquad p + q = r + s.$

Thus the coefficients are the same in

$$a_4a_0 - 4a_3a_1 + 3a_2a_2, \quad \text{and} \quad a_5a_1 - 4a_4a_2 + 3a_3a_3.$$

As an illustration of the preceding formula, I give the condition (the only one of the second degree) that an equation of the $2n^{\text{th}}$ degree should have $n+1$ equal roots, viz.

$$a_{2n}a_0 - 2na_{2n-1}a_1 + 2n\frac{(2n-1)}{2}a_{2n-2}a_2 - \&c. = 0,$$

where the coefficients are those of the binomial whose index is $2n$, with the exception that the coefficient of the last term in this equation is *half* the middle term of the binomial. This equation has been already given by Mr. Cayley (Note sur les Hyperdeterminantes, *Crelle's Journal*, vol. xxxiv.).

We do not obtain so simple a formula for the condition that an equation of the degree $2n+1$ should have $n+2$ equal roots, viz.

$$a_{2n+1}a_0 - (2n-1)a_{2n}a_1 + \frac{2n.(2n-3)}{1.2}a_{2n-1}a_2 - \frac{2n.(2n-1)(2n-5)}{1.2.3}a_{2n-2}a_3$$
$$+ \frac{2n(2n-1)(2n-2)(2n-7)}{1.2.3.4}a_{2n-3}a_4 - \&c. = 0.$$

I pass now to equations of the third degree in the coefficients, which must be of the form

$$\Sigma La_pa_qa_r = 0,$$

and after the substitution every term will be multiplied by the same power of a, if we have $p+q+r$ the same for every term. The substitution being equivalent to substituting for a_p, $(p+x)(p+y)$, &c. to k factors, &c. will give, if we denote $p+q+r$ by P, $pq+qr+rp$ by Q, and pqr by R, an equation of the form

$$\Sigma L(x^3+x^2P+xQ+R)(y^3+y^2P+yQ+R)(z^3+z^2P+zQ+R)\,\&c. = 0,$$

and will be satisfied identically if we have

$$\Sigma L=0,\ \Sigma LQ=0,\ \Sigma LR=0,\ \Sigma LQ^2=0,\ \Sigma LQR=0,\ \Sigma LR^2=0,\ \&c.$$

to $\quad \Sigma LQ^k = 0, \quad \Sigma LQ^{k-1}R = 0, \ldots . \Sigma LR^k = 0.$

And so in general the formation of an equation of any order $\Sigma La_pa_qa_r$, &c. $= 0$ depends on the solution of the system of equations

$$\Sigma L = 0, \quad \Sigma LQ = 0, \quad \Sigma LR = 0, \quad \Sigma LS = 0, \ \&c.,$$
$$\Sigma LQ^2 = 0, \quad \Sigma LQR = 0, \ \&c.,$$
$$\Sigma LQ^3 = 0, \ \&c.$$

where
$$Q = pq + qr + rp + \&c.,$$
$$R = pqr + pqs + \&c.,$$
$$S = pqrs + \&c.:$$

and where $P = p + q + r + \&c.$ is the same for all the terms.

It is often however more convenient, instead of solving these equations directly, to avail ourselves of equations of lower degree already formed.

Thus, let it be required to form an equation of the fourth degree in the coefficients which shall be satisfied when

$$a_5t^5 + 5a_4t^4 + 10a_3t^3 + 10a_2t^2 + 5a_1t + a_0 = 0$$

has three equal roots. Such an equation must of course be satisfied if the equation have four equal roots, and therefore must be of the form,

$$LA + MB + NC = 0;$$

where LMN are functions of the second degree in the coefficients, and $A, B, C,$ have the same signification as in Mr. Cayley's memoir.

Now if we substitute as before for a_p,

$$a^{n-k-p}(Ap^k + Bp^{k-1} + \&c.)$$

in several functions which fulfil the conditions that the equation should have $(n - k + 1)$ equal roots, viz. $\Sigma Lp^{k-1}q^{k-1} = 0$, &c., the results must be proportional to the $\Sigma p_k q_k$ for each function. In the case of $A, B, C,$ this is for each the same numerical factor, viz. 24. We may therefore attend only to the effect of the substitution on L, M, N. It is easy to see then that the coefficients of L, M, N must fulfil the conditions

$$\Sigma Lp^2q^2 = 0, \quad \Sigma Lpq(p+q) = 0, \quad \Sigma L.pq = 0,$$
$$\Sigma L.(p+q)^2 = 0, \quad \Sigma L(p+q) = 0, \quad \Sigma L = 0.$$

We secure the fulfilment of the last three by making the sum of the coefficients $= 0$ in L, M, N separately; since in each of these $p + q$ is the same: the remainder suffice for the determination of the problem.

Thus the function of the fourth degree for which the sum of the subindices $= 8$, must be of the form

$$A\{La_2a_3 + L_1a_3a_1 - (L + L_1)a_4a_0\} + BM(a_2a_1 - a_3a_0)$$
$$+ CN(a_1a_1 - a_2a_0).$$

The coefficients are then determined by the equations

$$\Sigma p^2q^2 = 0, \quad \text{or } 16L + 9L_1 + 4M + N = 0,$$

$$\Sigma pq = 0, \quad \text{or } 4L + 3L_1 + 2M + N = 0,$$

$$\Sigma pq(p+q) = 0, \quad \text{or } 16L + 12L_1 + 6M + 2N = 0.$$

Hence $L = 0$, $L_1 = 1$, $M = -3$, $N = +3$.

There are some functions of higher degrees, however, which we can perceive without calculation will be satisfied when the equation

$$a_n t^n + n_{n-1} t^{n-1} + \tfrac{1}{2} n \,.\, n - 1 \,.\, a_{n-2} t^{n-2} + \text{\&c.} = 0$$

has $n - k$ equal roots.

For then we have the system of $n - k$ equations,

$$a_n t^{k+1} + (k+1)\, a_{n-1} t^k + \frac{(k+1)\,k}{1.2}\, a_{n-2} t^{k-1} + \text{\&c.}$$

$$a_{n-1} t^{k+1} + (k+1)\, a_{n-2} t^k + \text{\&c.}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$a_{k+1} t^{k+1} + (k+1)\, a_k t^k + \text{\&c.}$$

When, therefore, $n - k$ is not less than $k + 2$, we may from any $k + 2$ of these equations eliminate t^{k+1}, &c. as independent unknowns; the determinant so formed will be the condition required. Thus, the condition that an equation should have $n - 1$ equal roots, is expressed by any determinant

$$\left\{\begin{matrix} a_p, & a_{p+1}, & a_{p+2} \\ a_q, & a_{q+1}, & a_{q+2} \\ a_r, & a_{r+1}, & a_{r+2} \end{matrix}\right\}.$$

It might likewise be expressed by any determinant of the form

$$\left\{\begin{matrix} a_p, & a_{p+\alpha}, & a_{p+\beta} \\ a_q, & a_{q+\alpha}, & a_{q+\beta} \\ a_r, & a_{r+\alpha}, & a_{r+\beta} \end{matrix}\right\}.$$

Mr. Cayley has proved, that we can express the condition that an equation should have $n - 2$ equal roots, by the sum of two such determinants. For if in the determinant just written we substitute for a_p, $(p + x)(p + y)$, &c., the result can evidently be only of the second degree in each of p, q, r. But it vanishes for $p = q$, or $p = r$, or $q = r$. It must therefore be $(p - q)(q - r)(r - p)$, multiplied by a factor independent of p, q, r. If then we have two such determinants, having $p + q + r$ the same for both, and if, having

divided each by its $(p-q)(q-r)(r-p)$, we equate the quotes, the result will be satisfied when the equation has $n-2$ equal roots.

It is easy to see that in like manner, if we have two determinants, for which $p+q+r+$ &c. is the same, each of which is satisfied when the equation has p equal roots, and that having divided each by the product of the differences $(p-q)(p-r)(q-r)$, &c., we equate the quotes, the result will be satisfied when the equation has $p-1$ equal roots.

The methods explained above may of course be applied directly to finding the condition that the given equation should have two equal roots. We can write down all the possible terms $a_k a_l$, &c., where the number of factors is $2(n-1)$ and $k+l+$ &c. $=n(n-1)$. The coefficients are determined then by a series of linear equations with numerical coefficients. There are some considerations which so far reduce the number of unknowns, as probably to make the labour of elimination not greater by this method than by the ordinary methods. Thus, if in the condition that $a_n t^n + a_{n-1} t^{n-1}$, &c. should have two equal roots, we make $a_n = 0$, we shall have the condition that $a_{n-1} t^{n-1} +$ &c. should have two equal roots multiplied by the square of a_{n-1}. We can thus write down many of the terms for $a_n t^n +$ &c., *à priori*, if we know the resultant for $a_{n-1} t^{n-1} +$ &c. Again, the result of elimination will be unchanged if we interchange a_n and a_0, a_{n-1} and a_1, &c. The resultant will then contain a number of pairs of terms having the same coefficients.

Thus, let it be required to determine the resultant for

$$a_3 t^3 + 3a_2 t^2 + 3a_1 t^1 + a_0.$$

We may write down

$$3a_2 a_2 a_1 a_1 - 4a_2 a_2 a_2 a_0 - 4a_1 a_1 a_1 a_3 + Aa_3 a_2 a_1 a_0 + Ba_3 a_3 a_0 a_0.$$

The first of these terms are obtained from knowing the resultant for

$$3a_2 t^2 + 3a_1 t^1 + a_0 = 0 \quad \text{and} \quad a_3 t^2 + 3a_2 t + 3a_1 = 0.$$

A and B are then given by the equations

$\Sigma L = 0$ or $3-4-4+A+B=0$,

$\Sigma L(klm+lmn+mnk+nkl) = 0$ or $3+12-4\times 8-4\times 10+6A=0$.

G. S.

March 1850.

ON THE GENERAL EQUATIONS OF GEODESIC LINES AND LINES OF CURVATURE ON SURFACES.

By Benjamin Dickson, Fellow of Trinity College, Dublin.

1. If through any point x, y, z, there be drawn any two right lines, whose equations are

$$\frac{x-x'}{\cos\alpha}=\frac{y-y'}{\cos\beta}=\frac{z-z'}{\cos\gamma}\ \ldots\ldots\ldots\ldots(\alpha),$$

$$\frac{x-x'}{\cos\alpha'}=\frac{y-y'}{\cos\beta'}=\frac{z-z'}{\cos\gamma'}\ \ldots\ldots\ldots\ldots(\alpha'),$$

it may be easily seen, that the two other right lines drawn through the same point, lying in the same plane with the former two, and bisecting the angles between them, will be represented by the equations

$$\epsilon'\,\frac{x-x'}{\cos\alpha'+\cos\alpha}=\epsilon'\,\frac{y-y'}{\cos\beta'+\cos\beta}=\epsilon'\,\frac{z-z'}{\cos\gamma'+\cos\gamma}\ \ldots(\beta),$$

$$\epsilon\,\frac{x-x'}{\cos\alpha'-\cos\alpha}=\epsilon\,\frac{y-y'}{\cos\beta'-\cos\beta}=\epsilon\,\frac{z-z'}{\cos\gamma'-\cos\gamma}\ \ldots(\gamma);$$

and also, if θ be the angle which the two former lines, (α) and (α') make with each other, that

$$\epsilon'=2\cos\tfrac{1}{2}\theta\quad \epsilon=2\sin\tfrac{1}{2}\theta.$$

2. Suppose the right lines (α) and (α') to be two consecutive tangents to a curve of double curvature; then will (β) be the right line joining their points of contact with the curve, and (γ) will coincide in direction with the radius of curvature of the curve at the point of intersection of these tangents, (since it lies in their plane and is *ultimately* perpendicular to both). If the angles which the latter line makes with the axes of coordinates be λ, μ, ν, the cosines of these angles will be seen, by referring to the equations of the line (γ), to be proportional, respectively, to

$$\cos\alpha'-\cos\alpha,\quad \cos\beta'-\cos\beta,\quad \cos\gamma'-\cos\gamma,$$

that is, (since α', β', γ', are the consecutive values of α, β, γ,) to

$$d\cos\alpha,\quad d\cos\beta,\quad d\cos\gamma,$$

but we know that

$$\cos\alpha=\frac{dx}{ds}\quad \cos\beta=\frac{dy}{ds}\quad \cos\gamma=\frac{dz}{ds},$$

and hence we at once derive the well-known expressions,

which determine the position of the radius of curvature of a curve of double curvature, with respect to the coordinate axes; viz.

$$\varepsilon \cos \lambda = d\frac{dx}{ds}, \quad \varepsilon \cos \mu = d\frac{dy}{ds}, \quad \varepsilon \cos \nu = d\frac{dz}{ds},$$

$$\varepsilon = \sqrt{\left\{\left(d\frac{dx}{ds}\right)^2 + \left(d\frac{dy}{ds}\right)^2 + \left(d\frac{dz}{ds}\right)^2\right\}},$$

and since θ (the angle of contact), being indefinitely small may be taken for its sine,

$$2 \sin \tfrac{1}{2}\theta = \theta = \varepsilon.$$

3. Let $F = 0$ be the equation of any surface; we shall adopt the following notation throughout,

$$X = \frac{dF}{dx}, \quad Y = \frac{dF}{dy}, \quad Z = \frac{dF}{dz},$$

$$S = \sqrt{(X^2 + Y^2 + Z^2)},$$

$$E = \sqrt{\left\{\left(d\frac{X}{S}\right)^2 + \left(d\frac{Y}{S}\right)^2 + \left(d\frac{Z}{S}\right)^2\right\}}:$$

the quantity E is, evidently, the indefinitely small angle between two consecutive normals to the surface, and will, in general, have a different value for every consecutive point in the vicinity of x, y, z.

4. In whatever direction the element ds of a curve be drawn through the point x, y, z, the following equation obtains,

$$\frac{Xdx + Ydy + Zdz}{Sds} = 0:$$

differentiate this equation, and there results

$$\left(\frac{X}{S}d\frac{dx}{ds} + \frac{Y}{S}d\frac{dy}{ds} + \frac{Z}{S}d\frac{dz}{ds}\right) + \left(\frac{dx}{ds}d\frac{X}{S} + \frac{dy}{ds}d\frac{Y}{S} + \frac{dz}{ds}d\frac{Z}{S}\right) = 0,$$

which may be seen by referring to the values of ε and E to be equivalent to

$$\varepsilon \cos \phi + \mathrm{E} \cos \Phi = 0,$$

ϕ being the angle between the normal to the surface and the plane of two consecutive tangents, or the osculating plane of the curve; and Φ the angle which the tangent to the curve makes with a plane parallel to two consecutive normals to the surface along it.*

* At every point of a *geodesic* line $\phi = 0$; and at every point of a *line of curvature* $\Phi = 0$.

5. If ρ_1 be the radius of absolute curvature of a curve traced upon the surface $F = 0$, and ρ the radius of curvature of the normal section, since

$$\rho = \frac{\rho_1}{\cos\phi} = \frac{\pm ds}{\epsilon \cos\phi} = \frac{\mp ds}{E \cos\Phi},$$

by substituting in these the values obtained above for $E\cos\Phi$ and $\epsilon\cos\phi$, there results

$$\rho = \frac{ds}{\frac{dx}{ds} d\frac{X}{S} + \frac{dy}{ds} d\frac{Y}{S} + \frac{dz}{ds} d\frac{Z}{S}} = \frac{-ds}{\frac{X}{S} d\frac{dx}{ds} + \frac{Y}{S} d\frac{dy}{ds} + \frac{Z}{S} d\frac{dz}{ds}},$$

$$= \frac{Sds^2}{dXdx + dYdy + dZdz}.$$

6. To obtain the equations of a *geodesic line*, we have only to recollect that the normal to the surface at any point of it lies in the plane of two consecutive tangents to the curve, and bisects the angle between them: this condition, from (2), is expressed by the following equations,

$$\frac{X}{d\frac{dx}{ds}} = \frac{Y}{d\frac{dy}{ds}} = \frac{Z}{d\frac{dz}{ds}} = \frac{S}{\epsilon} \ldots\ldots (\mathrm{I}).$$

For a *line of curvature*, the normals to the surface at any two consecutive points upon it intersect, and the tangent to the curve (or the line joining these consecutive points) lies in the plane of the two normals, and makes equal angles with them both: this condition, from (2), is expressed by the equations

$$\frac{dx}{d\frac{X}{S}} = \frac{dy}{d\frac{Y}{S}} = \frac{dz}{d\frac{Z}{S}} = \frac{ds}{E} \ldots\ldots (\mathrm{II}).$$

7. From the equations of the preceding paragraph the following theorems immediately result.

(1) *If the tangents to a geodesic line make a constant angle with a fixed right line, the normals to the surface along it will all be parallel to a fixed plane, and conversely;* for if

$$a\frac{dx}{ds} + b\frac{dy}{ds} + c\frac{dz}{ds} = \text{a const.},$$

by differentiating there results

$$a\,d\frac{dx}{ds} + b\,d\frac{dy}{ds} + c\,d\frac{dz}{ds} = 0:$$

and by substitution for $d\,\frac{dx}{ds}$, &c. from equations (I),

$$aX + bY + cZ = 0.$$

(2) *If the normals to a surface along a line of curvature make a constant angle with a fixed right line, the tangents to the curve will be all parallel to a fixed plane; that is, the curve will be a plane curve, and conversely:* for if

$$a\frac{X}{S} + b\frac{Y}{S} + c\frac{Z}{S} = \text{a const.},$$

then also $$ad\frac{X}{S} + bd\frac{Y}{S} + cd\frac{Z}{S} = 0;$$

and by substitution for $d\frac{X}{S}$, &c. from equations (II),

$$a\,dx + b\,dy + c\,dz = 0.$$

(3) *If the intersection of any two surfaces* ($F = 0$, $F_1 = 0$) *be a line of curvature on* both, *the surfaces will intersect throughout at a constant angle;* for at every such point the two following systems of equations are true,

$$\frac{dx}{d\frac{X}{S}} = \frac{dy}{d\frac{Y}{S}} = \frac{dz}{d\frac{Z}{S}},$$

$$\frac{dx}{d\frac{X_1}{S_1}} = \frac{dy}{d\frac{Y_1}{S_1}} = \frac{dz}{d\frac{Z_1}{S_1}};$$

and if we substitute for dx, dy, dz, from the first of these systems in

$$\frac{X_1dx + Y_1dy + Z_1dz}{S_1} = 0,$$

and from the second in

$$\frac{Xdx + Ydy + Zdz}{S} = 0,$$

and add, there will result

$$\frac{X}{S}d\frac{X_1}{S_1} + \frac{Y}{S}d\frac{Y_1}{S_1} + \frac{Z}{S}d\frac{Z_1}{S_1} + \frac{X_1}{S_1}d\frac{X}{S} + \frac{Y_1}{S_1}d\frac{Y}{S} + \frac{Z_1}{S_1}d\frac{Y}{S} = 0,$$

and therefore by integration

$$\frac{XX_1 + YY_1 + ZZ_1}{SS_1} = \text{a const.}$$

This theorem includes, as a particular case, the following one, demonstrated by Dr. Joachimsthal:* *If a line of curvature traced upon a surface be a plane curve, the tangent plane to the surface along it makes a constant angle with its plane.*

A theorem, to a certain extent analogous, is true for geodesic lines, viz.

(4) *If the intersection of any two surfaces* ($F=0$, $F_1=0$) *be a geodesic line upon both, the surfaces will intersect throughout at a constant angle:* for if we multiply together the equations expressing the conditions, that the curve whose coordinates are x, y, z should be a geodesic line both upon $F=0$ and upon $F_1=0$, and add, we obtain

$$\frac{XX_1+YY_1+ZZ_1}{SS_1}=\frac{1}{\varepsilon^2}\left\{\left(d\,\frac{dx}{ds}\right)^2+\left(d\,\frac{dy}{ds}\right)^2+\left(d\,\frac{dz}{ds}\right)^2\right\}.$$

If the angle of contact of the curve be *different from zero at every point,* the right-hand member of this equation becomes unity, and the surfaces will intersect throughout at an angle $=0$. But if the angle of contact *be zero at every point*, the right-hand member becomes $\frac{0}{0}$, and the angle at which the surfaces intersect is therefore indeterminate.

8. The two following equations are identically true for every element of a curve traced upon the surface $F=0$:

$$\frac{X}{S}\,d\,\frac{X}{S}+\frac{Y}{S}\,d\,\frac{Y}{S}+\frac{Z}{S}\,d\,\frac{Z}{S}=0,$$

$$\frac{dx}{ds}\,d\,\frac{dx}{ds}+\frac{dy}{ds}\,d\,\frac{dy}{ds}+\frac{dz}{ds}\,d\,\frac{dz}{ds}=0.$$

If in the first of these there be substituted for X, Y, and Z, the quantities to which they are proportional in equations (I.); and similarly in the second, for dx, dy, and dz, the quantities to which they are proportional in equations (II.); the resulting equation will be the same in both cases, viz.

$$d\,\frac{dx}{ds}\,d\,\frac{X}{S}+d\,\frac{dy}{ds}\,d\,\frac{Y}{S}+d\,\frac{dz}{ds}\,d\,\frac{Z}{S}=0\ldots\ldots\text{(III.)},\dagger$$

* *Crelle's Journal*, tom. xxx. p. 347.

† This equation is virtually the same with that obtained by Dr. Joachimsthal, in *Crelle's Journal*, tom. xxvi. p. 150, which he has written $VW=0$. In general, if p be the perpendicular let fall from the centre of absolute curvature of *any* curve upon the normal section, then will $V=p\,\varepsilon^2 S ds$, and if P be the *minimum* distance between two normals to a surface at consecutive points of *any* curve traced upon it, then will $W=P.ES^2$. And since for geodesic lines $p=0$ and for lines of curvature $P=0$, the differential equations of these lines are respectively $V=0$, $W=0$.

and as the process by which it was arrived at in the first instance shews that it must be true for every element of *a geodesic line*, so does that by which it was arrived at in the second instance prove that it is also true for every element of a *line of curvature.*

9. If the differentiation be performed, this equation will become

$$\frac{dS}{S} - \frac{d^2s}{ds} + \frac{dXd^2x + dYd^2y + dZd^2z}{dXdx + dYdy + dZdz} = 0 \ldots\text{(IV.)}.$$

By differentiating the expression in (5) for the radius of curvature of any curve traced upon a surface, we obtain the following:

$$\frac{d\rho}{\rho} = \frac{dS}{S} + 2\frac{d^2s}{ds} - \frac{dXd^2x + dYd^2y + dZd^2z + dxd^2X + dyd^2Y + dzd^2Z}{dXdx + dYdy + dZdz};$$

eliminating $\dfrac{d^2s}{ds}$ between these two, there results

$$\frac{d\rho}{\rho} - 3\frac{dS}{S} = \frac{dXd^2x + dYd^2y + dZd^2z}{dXdx + dYdy + dZdz} - \frac{dxd^2X + dyd^2Y + dzd^2Z}{dxdX + dydY + dzdZ} \quad \ldots\text{(V.)}.$$

And by this equation can be determined the value of the radius of curvature of the normal section at any point, either of a geodesic line or of a line of curvature, traced upon a surface.

10. In the particular case of surfaces *of the second order*, the right-hand member of equation (V.) is *identically* equal to zero; and therefore at any point of a geodesic line or of a line of curvature traced upon these surfaces

$$\rho = C.S^3.$$

As this subject however, for surfaces of the second order, has been elaborately discussed by several eminent geometers, we shall pursue it no farther than to remark than many of the properties demonstrated by them, and now familiarly known, result from the simple consideration that the constant C retains the *same* value for every point of a line of curvature and of *every* geodesic line which touches it.

Trinity College, Dublin, March 21, 1850.

NOTES ON MOLECULAR MECHANICS.

I.—ON THE GENERAL EQUATIONS OF MOTION.

By the Rev. Samuel Haughton.

The equations which express the conditions of equilibrium and motion resulting from molecular forces have been investigated by many writers, and although the results arrived at have for the most part agreed with each other, yet the principles from which they have been derived have been so different, as to render this branch of mechanics less complete than many others on which less labour has been bestowed. In France, this subject has occupied the attention of Navier, Poisson, Cauchy, and St. Venant, while at home it has been cultivated with success by Mr. Green and Prof. MacCullagh in the case of Light, and by Mr. Stokes in the case of Hydrodynamics and Elastic Solids. The principal cause of the slow progress of this as compared with other departments of dynamics seems to be the difficulty of comparing the results of theory with direct observations. In the case of other laws of dynamics, the law is presented immediately by observation, while in the case of molecular forces we can only observe the secondary results and effects of the forces, and these may often be produced by different mechanisms, so that we are at a loss to determine the precise method which nature has adopted.

My object in the present series of Notes on Molecular Mechanics is to shew that a degree of vagueness must attach to inquiries of this kind, particularly when applied to the phenomena of Light, which have been assumed rather than proved to belong to this department of mechanics. With this view I shall endeavour to establish a general system of equations from which the various theories proposed may be deduced by introducing hypotheses more or less probable. Having deduced and explained the different theories to some extent, I purpose to compare them, in the case of light, with the observed phenomena, and to shew that the evidence afforded by observation is not conclusive in favour of any theory which has been yet proposed.

I shall adopt the method of the *Mécanique Analytique* of Lagrange, which is well adapted to such investigations as the present. In order to express by means of it the conditions of equilibrium and motion of a continuous body, it is necessary to distinguish the forces acting at each point into two classes, viz. molecular and external forces; including among the latter the resultants of the attractions of the points of the body which are situated at a finite distance, as these attrac-

tions result from gravitation, and should not be confounded with molecular forces. The forces being thus considered as divided into two groups, the equation of virtual velocities is the following:

$$\Sigma(P\delta p + P'\delta p' + \&c.\dots) + \Sigma(Q\delta q + Q'\delta q' + \&c.) = 0,$$

P, P', &c. denoting the external forces, and Q, Q', &c. the molecular forces. The hypotheses which I shall make as to the nature of the molecular forces are two in number: first, '*that the virtual moments of the molecular forces may be represented by the variation of a single function,*' i.e.

$$Q\delta q + Q'\delta q' + \&c. = \delta V \dots\dots (1);$$

and secondly, '*that if* ξ, η, ζ *represent the small displacements of any molecule from its position of rest,* x, y, z, *the function* V *depends on the differential coefficients of the first order, of* ξ, η, ζ;' i.e.

$$V = F(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3) \dots\dots (2);$$

where
$$\alpha_1 = \frac{d\xi}{dx}, \quad \alpha_2 = \frac{d\xi}{dy}, \quad \alpha_3 = \frac{d\xi}{dz},$$
$$\beta_1 = \frac{d\eta}{dx}, \quad \beta_2 = \frac{d\eta}{dy}, \quad \beta_3 = \frac{d\eta}{dz},$$
$$\gamma_1 = \frac{d\zeta}{dx}, \quad \gamma_2 = \frac{d\zeta}{dy}, \quad \gamma_3 = \frac{d\zeta}{dz}.$$

Introducing the expression (1) into the equation of virtual velocities, and integrating by parts, we obtain

$$\left.\begin{aligned}
\iiint \epsilon \left(\frac{d^2\xi}{dt^2}\delta\xi + \frac{d^2\eta}{dt^2}\delta\eta + \frac{d^2\zeta}{dt^2}\delta\zeta\right) dx\,dy\,dz &= \iiint \delta V\, dx\,dy\,dz \\
= \iint\left(\frac{dV}{d\alpha_1}\delta\xi + \frac{dV}{d\beta_1}\delta\eta + \frac{dV}{d\gamma_1}\delta\zeta\right) dy\,dz & \\
+ \iint\left(\frac{dV}{d\alpha_2}\delta\xi + \frac{dV}{d\beta_2}\delta\eta + \frac{dV}{d\gamma_2}\delta\zeta\right) dx\,dz & \\
+ \iint\left(\frac{dV}{d\alpha_3}\delta\xi + \frac{dV}{d\beta_3}\delta\eta + \frac{dV}{d\gamma_3}\delta\zeta\right) dx\,dy & \\
- \iiint\left(\frac{d}{dx}\cdot\frac{dV}{d\alpha_1} + \frac{d}{dy}\cdot\frac{dV}{d\alpha_2} + \frac{d}{dz}\cdot\frac{dV}{d\alpha_3}\right)\delta\xi\, dx\,dy\,dz & \\
- \iiint\left(\frac{d}{dx}\cdot\frac{dV}{d\beta_1} + \frac{d}{dy}\cdot\frac{dV}{d\beta_2} + \frac{d}{dz}\cdot\frac{dV}{d\beta_3}\right)\delta\eta\, dx\,dy\,dz & \\
- \iiint\left(\frac{d}{dx}\cdot\frac{dV}{d\gamma_1} + \frac{d}{dy}\cdot\frac{dV}{d\gamma_2} + \frac{d}{dz}\cdot\frac{dV}{d\gamma_3}\right)\delta\zeta\, dx\,dy\,dz &
\end{aligned}\right\} \dots\dots (3),$$

where ϵ denotes the density of the body in the state of rest, and the double integrals belong to the conditions at the limits. The equations of motion deduced from (3) are the following:

$$\left.\begin{aligned}
-\epsilon\frac{d^2\xi}{dt^2} &= \frac{d}{dx}\cdot\frac{dV}{d\alpha_1}+\frac{d}{dy}\cdot\frac{dV}{d\alpha_2}+\frac{d}{dz}\cdot\frac{dV}{d\alpha_3}\\
-\epsilon\frac{d^2\eta}{dt^2} &= \frac{d}{dx}\cdot\frac{dV}{d\beta_1}+\frac{d}{dy}\cdot\frac{dV}{d\beta_2}+\frac{d}{dz}\cdot\frac{dV}{d\beta_3}\\
-\epsilon\frac{d^2\zeta}{dt^2} &= \frac{d}{dx}\cdot\frac{dV}{d\gamma_1}+\frac{d}{dy}\cdot\frac{dV}{d\gamma_2}+\frac{d}{dz}\cdot\frac{dV}{d\gamma_3}
\end{aligned}\right\}\ \ldots (4).$$

The limiting conditions which we shall have occasion to consider, are those which arise from two bodies in contact with each other; they are readily formed from (3) as follows: Let $F(x, y, z) = 0$ be the surface which passes through the positions of rest of the molecules which constitute the bounding surface; then the double integrals in (3) may be represented by

$$\begin{aligned}
\Delta = &\iint\left(\frac{dV}{d\alpha_1}\cdot\frac{dF}{dx}+\frac{dV}{d\alpha_2}\cdot\frac{dF}{dy}+\frac{dV}{d\alpha_3}\cdot\frac{dF}{dz}\right)\omega\,\delta\xi,\\
&+\iint\left(\frac{dV}{d\beta_1}\cdot\frac{dF}{dx}+\frac{dV}{d\beta_2}\cdot\frac{dF}{dy}+\frac{dV}{d\beta_3}\cdot\frac{dF}{dz}\right)\omega\,\delta\eta,\\
&+\iint\left(\frac{dV}{d\gamma_1}\cdot\frac{dF}{dx}+\frac{dV}{d\gamma_2}\cdot\frac{dF}{dy}+\frac{dV}{d\gamma_3}\cdot\frac{dF}{dz}\right)\omega\,\delta\zeta.
\end{aligned}$$

The equation (3) belonging to the second body will produce a similar set of terms: and since $F(x, y, z)$ is common to both bodies, and ξ, η, ζ are the same for both, we obtain finally

$$\left.\begin{aligned}
\left(\frac{dV_0'}{d\alpha_1}-\frac{dV_0''}{d\alpha_1}\right)\frac{dF}{dx}+\left(\frac{dV_0'}{d\alpha_2}-\frac{dV_0''}{d\alpha_2}\right)\frac{dF}{dy}+\left(\frac{dV_0'}{d\alpha_3}-\frac{dV_0''}{d\alpha_3}\right)\frac{dF}{dz}=0,\\
\left(\frac{dV_0'}{d\beta_1}-\frac{dV_0''}{d\beta_1}\right)\frac{dF}{dx}+\left(\frac{dV_0'}{d\beta_2}-\frac{dV_0''}{d\beta_2}\right)\frac{dF}{dy}+\left(\frac{dV_0'}{d\beta_3}-\frac{dV_0''}{d\beta_3}\right)\frac{dF}{dz}=0\\
\left(\frac{dV_0'}{d\gamma_1}-\frac{dV_0''}{d\gamma_1}\right)\frac{dF}{dx}+\left(\frac{dV_0'}{d\gamma_2}-\frac{dV_0''}{d\gamma_2}\right)\frac{dF}{dy}+\left(\frac{dV_0'}{d\gamma_3}-\frac{dV_0''}{d\gamma_3}\right)\frac{dF}{dz}=0
\end{aligned}\right\}\ldots(5),$$

V_0 signifying that we have substituted in V the values of x, y, z, which satisfy the condition $F(x, y, z) = 0$.

To these equations (5) should be added the geometrical equations which express that the vibrating molecules at the

bounding surface may be considered as belonging to either body; they are three in number,

$$\xi_0' = \xi_0'', \quad \eta_0\eta' = {}_0''; \quad \zeta_0' = \zeta_0'' \ldots\ldots\ldots (6).$$

Equations (4), (5), (6) are necessary and sufficient to determine the laws of propagation, reflexion, and refraction of waves, and shew that a connexion must exist between them; consequently, no mechanical theory of vibrations can be correct which does not exhibit this connexion, or which assumes such laws of reflexion and refraction as contradict the laws of propagation. A connexion between these laws is no proof of the truth of a theory, but the want of such a connexion would be a proof of the inconsistency of a theory.

It may be useful to compare the results we have arrived at with the ordinary method of deducing the equations of molecular equilibrium. Let us conceive an elementary parallelepiped of the body, one corner of which is situated at the point x, y, z, (which here denote the actual position of the molecule) the sides of the parallelepiped being dx, dy, dz; if the pressures exerted on the parallelepiped by the surrounding parts of the body be considered as oblique (in Hydrostatics they are considered as normal), and their components be represented by the following notation,

P_1, Q_1, R_1 components parallel to x, y, z of pressure on face $dydz$,
P_2, Q_2, R_2 .... $dxdz$,
P_3, Q_3, R_3 .... $dxdy$,

then will the pressure on the sides opposite to these be

$$P_1 + \frac{dP_1}{dx}\,dx, \quad Q_1 + \frac{dQ_1}{dx}\,dx, \quad R_1 + \frac{dR_1}{dx}\,dx,$$

$$P_2 + \frac{dP_2}{dy}\,dy, \quad Q_2 + \frac{dQ_2}{dy}\,dy, \quad R_2 + \frac{dR_2}{dy}\,dy,$$

$$P_3 + \frac{dP_3}{dz}\,dz, \quad Q_3 + \frac{dQ_3}{dz}\,dz, \quad R_3 + \frac{dR_3}{dz}\,dz:$$

there must be equilibrium between these forces, resulting from molecular action, and the external forces acting on the parallelepiped; hence finally we obtain

$$\left.\begin{aligned} \rho\left(X - \frac{d^2x}{dt^2}\right) &= \frac{dP_1}{dx} + \frac{dP_2}{dy} + \frac{dP_3}{dz} \\ \rho\left(Y - \frac{d^2y}{dt^2}\right) &= \frac{dQ_1}{dx} + \frac{dQ_2}{dy} + \frac{dQ_3}{dz} \\ \rho\left(Z - \frac{d^2z}{dt^2}\right) &= \frac{dR_1}{dx} + \frac{dR_2}{dy} + \frac{dR_3}{dz} \end{aligned}\right\} \ldots\ldots (7),$$

where ρ denotes the density at the point x, y, z; these equations will reduce to (4), on the supposition that external forces are wanting, and that x, y, z are the coordinates of the position of rest of the molecule.

By comparing equations (4) and (7), we can assign a physical meaning to the differential coefficients of V with respect to the variables a_1, a_2, &c.; viz.

$$\left.\begin{array}{lll} \dfrac{dV}{da_1} = P_1, & \dfrac{dV}{da_2} = P_2, & \dfrac{dV}{da_3} = P_3 \\ \dfrac{dV}{d\beta_1} = Q_1, & \dfrac{dV}{d\beta_2} = Q_2, & \dfrac{dV}{d\beta_3} = Q_3 \\ \dfrac{dV}{d\gamma_1} = R_1, & \dfrac{dV}{d\gamma_2} = R_2, & \dfrac{dV}{d\gamma_3} = R_3 \end{array}\right\} \ldots\ldots (8).$$

We can pass, by the aid of these equations, from the physical conception of a body to the molecular function V, which is peculiar to it; and by giving different forms to the function V, we may obtain as many different kinds of elastic media as we please: but as several important properties, common to all, may be deduced from equations (4), I purpose to discuss these in my next note, and then to enter upon the consideration of special forms of V, which have been proposed by various writers for elastic solids and the assumed ether of the undulatory theory of light.

Trinity College, Dublin, March 28, 1850.

THEOREM ON THE QUADRATURE OF SURFACES.

By the Rev. John H. Jellett, A.M., Trinity College, Dublin.

Let tangent cones be drawn to any surface from all points of a given line, and let the side of any one of these cones be represented by T. Let θ, ϕ be the polar angles which determine the position of the corresponding side of the reciprocal cone, θ being measured from the given line. Then if we denote by C the superficial area of any one of these cones (bounded by the curve of contact), and by S the superficial area of the included part of the surface, we shall have

$$C - S = \tfrac{1}{2}\iint T^2 \sin\theta\, d\theta\, d\phi,$$

the double integral being taken through the whole of the enveloped portion of the surface.

I have before shewn (Calculus of Variations, p. 376), that the variation of the quantity

$$\iint(\tfrac{1}{2}T^2 \sin\theta\, d\theta\, d\phi + dS)$$

depends solely upon the variations of the limiting values of y, z, and p or q $\left(\text{where } p = \dfrac{dz}{dx},\ q = \dfrac{dz}{dy}\right)$.

Hence it is evident that if two surfaces be described touching along a closed curve, the value of the integral for the enclosed portion of each of these surfaces will be the same. Hence if a cone touch any surface along a closed curve, we shall have

$$\iint(\tfrac{1}{2}T^2 \sin\theta d\theta d\phi + dS) = \iint(\tfrac{1}{2}T'^2 \sin\theta' d\theta' d\phi' + dS'),$$

the accented letters referring to the cone. But since

$$\sin\theta' d\theta' d\phi' = \frac{r't' - s'^2}{(1 + p'^2 + q'^2)^{\frac{3}{2}}}\, dxdy,$$

it is plain that for the cone, which is a developable surface,

$$\iint\tfrac{1}{2}T'^2 \sin\theta' d\theta' d\phi' = 0.$$

Hence, denoting by C the superficial area of the cone,

$$\iint\tfrac{1}{2}T^2 \sin\theta d\theta d\phi = C - S.$$

This theorem may easily be verified for a surface of revolution, the tangent cone being drawn from a point in the axis. In certain cases a similar theorem may be found for a portion of the surface bounded by any closed curve, the cone being replaced by the developable surface which touches along the bounding curve.

ON A THEOREM IN CONFOCAL SURFACES OF THE SECOND ORDER.

By Richard Townsend.

The following elegant theorem was given some years ago by Mr. Salmon.

At the point of intersection of any three confocal surfaces of the second order, the two centres of curvature of each surface are the two poles of its own tangent plane with respect to the other two surfaces.

The following is a geometrical proof of this theorem, which, though not perhaps the simplest that might be given

for some of its particular cases, has the advantage of applying equally to both classes of surfaces, central and paraboloidal, and, when brought down from surfaces to curves, of holding also, with scarcely any modification, for any two confocal conics, central or non-central.

Denoting the three surfaces by $SS'S''$, and their point of intersection by O, let a point P be taken arbitrarily at any distance from O on the curve of intersection of any two of the surfaces, of S' suppose, and of S'', and at that point let two tangent planes $T'T''$ and two normals $N'N''$ be drawn to those surfaces; let also a cone Σ be sent off from the same point enveloping the third surface S, and let its plane of contact Π be taken. This last plane will intersect the surface S in a conic s, the two tangent planes $T'T''$ in two right lines $L'L''$, and the two normals $N'N''$ in two points $P'P''$.

Now these two latter points $P'P''$ are those which in the limit when P is taken infinitely near to O become ultimately the two centres of curvature of S' and S'' corresponding to the element OP of their common line of curvature at O; for the plane Π in which they both lie, the polar of P with respect to S, becomes then ultimately the tangent plane to that surface at O, and therefore ultimately contains the two normals to the other two surfaces S' and S'' with the two elements of their second lines of curvature at the same point. Whatever property therefore is true of those two points P' and P'' in general when P is taken at any distance from O, will, suitably modified if necessary, be true also of the two aforesaid centres of curvature, these latter we see being in fact no other than the same two points $P'P''$ in the particular case when ultimately P is taken infinitely near to O.

But, *whatever be the distance of* P, P' and P'' *are always the two poles with respect to* S *of* T' *and* T''; for, from a fundamental property of confocal surfaces, the two planes $T'T''$ are two principal planes, and the two lines $N'N''$ the two corresponding principal axes of the enveloping cone Σ; therefore the two points $P'P''$ are the poles of the two lines $L'L''$ with respect to the conic s, and they also lie both in the polar plane of the point P with respect to the surface S.

Thus the theorem is proved for two of the six centres of curvature, and therefore for them all, a similar method being of course applicable to the remaining four.

Trinity College, Dublin, Nov. 29, 1849.

MATHEMATICAL NOTES.

I.—On a Theorem in Mr. Hearn's Paper (vol. IV. p. 265).

The theorem in Mr. Hearn's memoir in the *Mathematical Journal* (Nov. 1849), that the nine quantities contained in the nine equations

$$\left.\begin{aligned} a_1 + a_2 + a_3 &= 1 \\ b_1 + b_2 + b_3 &= 1 \\ c_1 + c_2 + c_3 &= 1 \end{aligned}\right\} \ldots\ldots\ldots\ldots (1),$$

$$\left.\begin{aligned} a_1^2 + a_2^2 + a_3^2 &= 1 \\ b_1^2 + b_2^2 + b_3^2 &= 1 \\ c_1^2 + c_2^2 + c_3^2 &= 1 \end{aligned}\right\} \ldots\ldots\ldots\ldots (2),$$

$$\left.\begin{aligned} b_1c_1 + b_2c_2 + b_3b_3 &= 0 \\ c_1a_1 + c_2a_2 + c_3a_3 &= 0 \\ a_1b_1 + a_2b_2 + a_3b_3 &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (3),$$

are not absolutely determined by those equations, but that if one of them be given the rest are known, may be readily proved without any reference to solid geometry.

Suppose $a_1 + a_2 + a_3 = h$, the remaining eight equations being left as above. By squaring this last and the other two equations (1), adding the results together, and properly arranging the sum, we get the equation

$$a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 + c_1^2 + c_2^2 + c_3^2 + 2(a_2a_3 + b_2b_3 + c_2c_3)$$
$$+ 2(a_3a_1 + b_3b_1 + c_3c_1) + 2(a_1a_2 + b_1b_2 + c_1c_2) = h^2 + 2.$$

The nine squares on the left-hand side of this equation are together equal to 3, as appears from (2). Each of the three sets of products vanishes, as is shewn by the well-known transformation of the equations (2) and (3). Thus this equation is reduced to $3 = h^2 + 2$, or $h = \pm 1$. This shews that, though the equation $a_1 + a_2 + a_3 = 1$ is not a necessary consequence of the other eight equations, for $a_1 + a_2 + a_3 = -1$ would equally satisfy them; yet it does not give such an independent relation between the unknown quantities as would make it, together with the other eight equations, sufficient for determining the nine quantities.

If only seven of the nine equations be assumed, and $a_1 + a_2 + a_3 = h$, and $b_1 + b_2 + b_3 = k$, the only equation which we can find for determining h and k is $h^2 + k^2 = 2$, which leaves them indeterminate; so that an eighth equation does give a relation between the quantities which is independent of the other seven.

Thus we see that (1), (2), and (3) consist of eight independent equations, from which, if one of the quantities be given, the other eight are known.

A theorem somewhat more general may be proved in like manner, viz. that if the equations (2) and (3) hold, and

$$pa_1 + qa_2 + ra_3 = h,$$
$$pb_1 + qb_2 + rb_3 = k,$$

where p, q, r, h, and k have any values whatsoever, then

$$pc_1 + qc_2 + rc_3 = l,$$

where $p^2 + q^2 + r^2 = h^2 + k^2 + l^2$.

It is easy to see that a geometrical theorem correlative with this last is, that if a sphere be cut by two parallel planes, and on the two circles of intersection respectively be taken any two points P, Q, such that the arc PQ may be a quadrant, and another plane parallel to the former two be taken (on either side of the centre) such that the sum of the squares of the distances of these three planes from the centre may be equal to the square of the radius of the sphere, this last plane will contain the vertex of a quadrantal triangle whose base is PQ.

L. J. F.

II.—Théorèmes sur l'Intégration de l'Equation

$$\frac{d^2y}{dx^2} + \frac{r}{x}\frac{dy}{dx} = \left(bx^m + \frac{s}{x^2}\right)y.$$

Par C. J. Malmstén.

On sait qu'il y a deux équations différentielles du 2e ordre

$$\frac{d^2y}{dx^2} + bx^my = 0\ldots.(1) \quad \text{et} \quad \frac{d^2y}{dx^2} + \frac{r}{x}\cdot\frac{dy}{dx} + by = 0\ldots.(2),$$

qui appartiennent au genre de Riccati, c. à d. dont l'intégration pourra s'effectuer par des quadratures indéfinies toutes les fois que m ou r satisfait à certaines conditions et ces conditions sont

(α) *pour l'équation* (1) que $m = -\dfrac{4n}{2n \pm 1}$,

(β) *pour l'équation* (2) que r = un nombre pair, positif ou negatif.

Cependant ces deux équations ne sont en effet que des cas très spéciaux de l'équation beaucoup plus générale

$$\frac{d^2y}{dx^2} + \frac{r}{x} \cdot \frac{dy}{dx} = \left(bx^m + \frac{s}{x^2}\right) y \ldots\ldots\ldots\ldots (3),$$

pour l'intégration de laquelle par quadratures indéfinies le théorème suivant contient un criterium aussi simple que générale.

Théorème *Pour l'équation différentielle linéaire du* 2^e^ *ordre*

$$\frac{d^2y}{dx^2} + \frac{r}{a} \cdot \frac{dy}{dx} = \left(bx^m + \frac{s}{x^2}\right) y \ldots\ldots\ldots\ldots (3),$$

soit intégrable au moyen de quadratures indéfinies relatives à x, *il est nécessaire et il suffit, qu'entre* r, m, *et* s *cette relation ait lieu*

$$m + 2 = \pm \frac{2\sqrt{\{(1-r)^2 + 4s\}}}{2n+1},$$

étant n *un nombre entier quelconque ou zéro.*

Coroll. 1. Pour s = 0, c. à d. pour l'équation

$$\frac{d^2y}{dx^2} + \frac{r}{a} \cdot \frac{dy}{dx} = bx^m y,$$

la condition d'intégrabilité sera

$$m = -\frac{4\left(n \pm \frac{1}{2}r\right)}{2n \pm 1},$$

ce qui pour $r = 0$ et $m = 0$ redonne les conditions (α) et (β). Pour $m = -1$, on aura $\quad r = \pm 2k,$

où k est un nombre inpair quelconque, pour condition d'intégrabilité de l'equation

$$x \frac{d^2y}{dx^2} + r \cdot \frac{dy}{dx} = by,$$

qui est précisement le cas, où la méthode de Mr. Liouville (méthode de différentiation à une indice quelconque) est en défaute.

Coroll. 2. Pour s = r, c. à d. pour l'équation

$$\frac{d^2y}{dx^2} + \frac{r}{x} \cdot \frac{dy}{dx} = \left(bx^m + \frac{r}{x^2}\right) y,$$

la condition d'intégrabilité sera

$$m = -\frac{4\left(n \pm \frac{1}{2}r\right)}{2n \mp 1}.$$

Coroll. 3. Pour r=1, c. à d. pour l'équation (en posant $s=a^2$)

$$\frac{d^2y}{dx^2}+\frac{1}{x}\cdot\frac{dy}{dx}=\left(bx^m+\frac{a^2}{x^2}\right)y,$$

on aura, pour condition d'intégrabilité,

$$m=-4\cdot\frac{n\mp(a-\frac{1}{2})}{2n\pm1}.$$

Coroll. 4. Pour r = 0, c. à d. pour l'équation $\{$en posant $s=a(a+1)\}$

$$\frac{d^2y}{dx^2}=\left\{bx^m+\frac{a(a+1)}{x^2}\right\}.y,$$

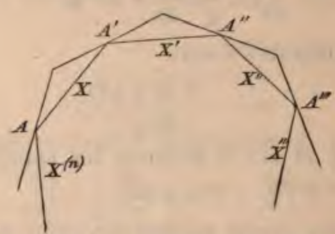
la condition d'intégrabilité sera

$$m=-\frac{4(n\mp a)}{2n\pm1}.$$

No. III.—Analytical Theorem Concerning Polygons.

By William Walton.

The sides $A, A', A'', A''',\dots$ of any polygon, and the sides $X, X', X'', X''',\dots$ of any polygon inscribed within



it, may be represented by the system of equations

$$\left|\begin{matrix}A\\u=0\end{matrix}\right|,\quad\left|\begin{matrix}A'\\u'=0\end{matrix}\right|,\quad\left|\begin{matrix}A''\\u''=0\end{matrix}\right|,\quad\left|\begin{matrix}A'''\\u'''=0\end{matrix}\right|\dots,\quad\left|\begin{matrix}A^{(n)}\\u^{(n)}=0\end{matrix}\right|;$$

$$X\dots au+u'+u''+u'''+\dots\dots+u^{(n)}=0,$$
$$X'\dots au+au'+u''+u'''+\dots\dots+u^{(n)}=0,$$
$$X''\dots au+au'+au''+u'''+\dots\dots+u^{(n)}=0,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$X^{(n)}\dots au+au'+au''+au'''+\dots\dots+au^{(n)}=0.$$

Let the sides $A, A', A'',\dots$ of the polygon be represented by the equations

$$A=0,\quad A'=0,\quad A''=0,\dots\dots$$

and the sides X, X', X'',...., by the equations

$$X = 0, \quad X' = 0, \quad X'' = 0, \ldots\ldots$$

Since A, X, $X^{(n)}$, pass through a single point,

$$X^{(n)} = \lambda A + \mu X,$$

λ and μ being constants. In like manner

$$\begin{aligned} X &= \lambda_1 A' + \mu_1 X', \\ X' &= \lambda_2 A'' + \mu_2 X'', \\ X'' &= \lambda_3 A''' + \mu_3 X''', \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \\ X^{(n-1)} &= \lambda_n A^{(n)} + \mu_n X^{(n)}. \end{aligned}$$

Multiply these $n + 1$ equations in order by

$$1, \ \mu, \ \mu\mu_1, \ \mu\mu_1\mu_2, \ldots \ \mu\mu_1\mu_2 \ldots \mu_{n-1};$$

then, representing

$$X^{(n)}, \ \mu X, \ \mu\mu_1 X', \ldots \ \mu\mu_1\mu_2 \ldots \mu_{n-1} X^{(n-1)},$$

by $\quad Y^{(n)}, \quad Y, \quad Y', \ldots\ldots\ldots\ldots\ldots \ Y^{(n-1)},$

and $\quad \lambda A, \ \mu\lambda_1 A', \ \mu\mu_1\lambda_2 A'', \ldots \ \mu\mu_1\mu_2 \ldots \mu_{n-1}\lambda_n A^{(n)},$

by $\quad u, \ u', \ u'', \ldots \ u^{(n)},$

we have, putting a for $\mu\mu_1\mu_2 \ldots \mu_{n-1}$,

$$\begin{aligned} Y^{(n)} &= u + Y, \\ Y &= u' + Y', \\ Y' &= u'' + Y'', \\ Y'' &= u''' + Y''', \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \\ Y^{(n-1)} &= u^{(n)} + a Y^{(n)}. \end{aligned}$$

From these equations we have

$$\begin{aligned} (1-a) Y^{(n)} &= u + u' + u'' + \ldots + u^{(n)}, \\ (1-a) Y &= au + u' + u'' + \ldots + u^{(n)}, \\ (1-a) Y' &= au + au' + u'' + \ldots + u^{(n)}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (1-a) Y^{(n-1)} &= au + au' + au'' + \ldots + au^{(n)}. \end{aligned}$$

Since u, u', u'',... are constant multiples of A, A', A'',... and the right-hand members of the last n equations are constant multiples of $Y^{(n)}$, Y, Y',...., and therefore of $X^{(n)}$, X, X',...., the truth of the proposition is established.

Ex. Having given the equations to the sides of any quadrilateral, to find the equations to the sides of an inscribed quadrilateral the opposite sides of which intersect in two given points.

Let the equations to the sides of the original quadrilateral be

$$k = 0, \quad l = 0, \quad m = 0, \quad n = 0.$$

Then, since u, u', u'', u''', must be some constant multiples of k, l, m, n, respectively, the equations to the sides of the inscribed quadrilateral will be of the forms

$$\alpha k + \beta l + \gamma m + \delta n = 0 \ldots\ldots\ldots\ldots (1),$$
$$\alpha (k + \beta l) + \gamma m + \delta n = 0 \ldots\ldots\ldots\ldots (2),$$
$$\alpha (k + \beta l + \gamma m) + \delta n = 0 \ldots\ldots\ldots\ldots (3),$$
$$k + \beta l + \gamma m + \delta n = 0 \ldots\ldots\ldots\ldots (4).$$

Let the values of k, l, m, n, be k_1, l_1, m_1, n_1, and k_2, l_2, m_2, n_2, at the given points.

Then we shall have, for the determination of the constants $\alpha, \beta, \gamma, \delta$,

$$\alpha k_1 + \beta l_1 + \gamma m_1 + \delta n_1 = 0 \ldots\ldots\ldots\ldots (5),$$
$$\alpha (k_2 + \beta l_2) + \gamma m_2 + \delta n_2 = 0 \ldots\ldots\ldots\ldots (6),$$
$$\alpha (k_1 + \beta l_1 + \gamma m_1) + \delta n_1 = 0 \ldots\ldots\ldots\ldots (7),$$
$$k_2 + \beta l_2 + \gamma m_2 + \delta n_2 = 0 \ldots\ldots\ldots\ldots (8).$$

From (5) and (7), we have

$$\beta l_1 + \gamma m_1 = 0, \quad \alpha k_1 + \delta n_1 = 0;$$

and, from (6) and (8),

$$k_2 + \beta l_2 = 0, \quad \gamma m_2 + \delta n_2 = 0.$$

Substituting the values of $\alpha, \beta, \gamma, \delta$, given by these four equations in (1), (2), (3), (4), we shall obtain, for the required equations to the sides of the inscribed quadrilateral,

$$\frac{n_1 k - k_1 n}{n_2 k_1} + \frac{l_1 m - m_1 l}{l_1 m_2} = 0,$$
$$\frac{k_2 l - l_2 k}{k_1 l_2} + \frac{m_2 n - n_2 m}{m_2 n_1} = 0,$$
$$\frac{l_1 m - m_1 l}{l_2 m_1} + \frac{n_1 k - k_1 n}{n_1 k_2} = 0,$$
$$\frac{m_2 n - n_2 m}{m_1 n_2} + \frac{k_2 l - l_2 k}{k_2 l_1} = 0.$$

December 26, 1849.

ON THE COMPLETE HEXAGON INSCRIBED IN A CONIC SECTION.

By the Rev. Thos. P. Kirkman, M.A.

§. I.—*Linear relations among the intersections of Pascal's lines.*

If we distribute the numerals 123456 in any order upon the six angles of a hexagon inscribed in a conic section $C=0$, and put $12=0$ for the equation of the line joining 1 and 2, &c., the following three equations will exhibit, if the constants $\lambda, \lambda_{,}, \lambda_{,,}$ be properly determined, three of Pascal's sixty lines (A):

$$12.34.56 - \lambda\, 23.45.61 = 0 = CA,$$

$$21.43.65 - \lambda_{,}\, 14.36.52 = 0 = CA_{,},$$

$$21.43.56 - \lambda_{,,} 14.35.62 = 0 = CA_{,,}.$$

At the intersection of the lines A and $A_{,}$ we have

$$\lambda 23.61.45 - \lambda_{,} 36.14.52 = 0 = CA';$$

whence we see that the three pascalians A, $A_{,}$, A' meet in a point. Since there is such a point in each of the 60 (A), we thus obtain the proof of Professor Steiner's theorem, (vide *Appendix to his Systematische Entwickelung u. s. w. Berlin*, 1832):

"*The sixty lines of Pascal go three together through twenty given points* G."

If we denote the line A, above exhibited, by the permutation 123456, which may be read backwards, or cyclically permuted, without changing its signification, we see that the point G just found is the intersection of any of the three pairs of pascalians,

$$A\,123456, \quad A_{,}143652, \quad A'163254,$$

$$A_{,}214365, \quad A'416325, \quad A\,612345;$$

and may be represented by the symbol $G12.34.56$, or by $G14.36.52$, or by $G16.32.54$; this symbol being unaltered in its meaning by cyclical permutation, either in its odd or in its even places. It is hence evident, that $G12.34.56$, $G21.43.65$, $G12.56.34$, are all the same point, the order of the three *syllables* 12, &c. being indifferent, if that of their elements 1 and 2, &c. be changed in none or in all the three.

At the intersection of A and $A_{,,}$, we have

$$\lambda 23.54.16 - \lambda_{,,} 35.41.62 = 0 = CA'';$$

whence it follows, that the three pascalians A, $A_{,,}$, A'' meet in a point (H), which is the intersection of any one of the three pairs,

$$A\ 123456,\quad A\ 456123,\quad A_{,,}356214,$$
$$A_{,,}214356,\quad A''541623,\quad A''532614;$$

and which may be denoted by any one of the three symbols,

$$H(12.34)\,56,\quad H(45.61)\,23,\quad H(35.62)\,14:$$

in which the bracketed syllables, and these only, may both (or neither) have their elements permuted, without changing the symbol. The line 123456, which is 345612 or 561234, will evidently contain the three distinct points

$$H(12.34)\,56,\quad H(34.56)\,12,\quad \text{and } H(56.12)\,34,$$

and can contain no more; since $H(45.61)\,23$, $H(61.23)\,45$, and $H(23.45)\,61$, are in order identical with those three. As every pascalian line has three of these points (H), we are enabled to add to Steiner's theorem above quoted the following

Theor. I. *The sixty lines of Pascal go not only three together through twenty points* (G), *but also three together through sixty points* (H), *so that in each pascalian line are a point* G *and three points* H.

It is evident that the three pascalians which meet in a point G or H are indicated by the symbols of the points. Thus, $G\,12.34.56$ is the intersection of the three 123456, 143652, 163254; and $H(12.34)\,56$ that of 123456, 541623, and 356214; or, more simply, as a deduction from the symbol, 456123, 541623, and 214356.

Let us consider the four points $G\,12.34.56$, $G\,21.34.56$, $G\,12.43.56$, $G\,12.34.65$, which are in order the intersections of the four pairs,

$$(a)\ 123456,\quad (e)\ 213456,\quad (o)\ 124356,\quad (u)\ 123465,$$
$$(a_{,})\ 214365,\quad (e_{,})\ 124365,\quad (o_{,})\ 213465,\quad (u_{,})\ 214356.$$

It is evident, putting a for the pascalian 123456, &c., that a and e meet at the intersection (p) of $12 = 0$ with $45 = 0$; or that

(ae) is (12.45), $(a_{,}e_{,})$ is (12.36), $(ou_{,})$ is (12.35), $(o_{,}u)$ is (12.46),
(ao) is (34.61), $(a_{,}o_{,})$ is (34.52), $(eu_{,})$ is (34.62), $(e_{,}u)$ is (34.51),
(au) is (56.23), $(a_{,}u_{,})$ is (56.14), $(eo_{,})$ is (56.13), $(e_{,}o)$ is (56.24).

If then λ be properly determined, we have, writing symbols of lines for their equations,

$$au_{,}o_{,}e_{,} - \lambda a_{,}uoe = 0 = 12\Theta = 12.34.56.I,$$

where $I = 0$ is a given line containing the four G-points $(aa_{,})$, $(ee_{,})$, $(oo_{,})$, and $(uu_{,})$. This line may be denoted by $I12.34.56$, where the three syllables 12, &c. may have any order, or their elements either. The number of such symbols that can be made is fifteen; and we have thus demonstrated the second proposition delivered by Steiner in the theorem already quoted:

"*The twenty points* (G), *in which meet three together the sixty lines of Pascal, lie four together on fifteen given lines* (I)."

Omitting now the subindex in $I12.34.56$, we have the equations

$$ae_{,} - \theta a_{,}e = 0 = I.12,$$
$$ao_{,} - \theta' a_{,}o = 0 = I.34,$$
$$eu - \theta'' e_{,}u_{,} = 0 = I.34;$$

whence follow the pair

$$\theta' oe_{,} - \theta o_{,}e = 0 = IJ,$$
$$\theta\theta'' a_{,}u_{,} - au = 0 = IJ'.$$

The point (12.34) is not on $I12.34.56$, which is equally $I12.56.34$, and can as easily be proved to contain (12.56) as (12.34): it is therefore the intersection of the two given lines $J = 0$ and $J' = 0$. The first of these lines contains the points (oe), or $H(12.43)\,56$, and $(o_{,}e_{,})$, or $H(12.43)\,65$; and may be denoted by the symbol $J12.43$: the second contains $(a_{,}u)$, or $H(12.34)\,65$, and $(au_{,})$, or $H(12.34)\,56$; and is completely defined by the symbol $J12.34$; for this indicates three points in it. It is easily seen that the order of the two syllables (12 and 34) is indifferent in this symbol.

Since (au), $(a_{,}u_{,})$, $(o_{,}e)$, and $(oe_{,})$ are points in $56 = 0$, we have

$$oo_{,} - \lambda ee_{,} = 0 = J.56,$$
$$aa_{,} - \lambda' uu_{,} = 0 = J'.56;$$

and also

$$J + kJ' = 0 = 12,$$
$$J + k_{,}J' = 0 = 34.$$

At $(u_{,}o)$ or $(uo_{,})$, both in $12 = 0$, we have alike

$$\frac{J}{J'} = -\frac{\lambda ee_{,}}{aa_{,}} = -k;$$

and at $(e_{,}u)$ or $(eu_{,})$, both in $34 = 0$,

$$\frac{J}{J'} = \frac{oo_{,}}{aa_{,}} = -k_{,}.$$

The value of λ is that of $oo_{,} : ee_{,}$ at 56.12, say $(oo_{,} : ee_{,})_{56.12}$; whence

$$-k = \frac{J}{J'} = -\left(\frac{oo_{,}}{ee_{,}} \cdot \frac{ee_{,}}{aa_{,}}\right)_{56.12} = -\left(\frac{oo_{,}}{aa_{,}}\right)_{56.12} = -\left(\frac{oo_{,}}{aa_{,}}\right)_{34.12} = k_{,},$$

which proves that $J12.43$ and $J12.34$ are synharmonicals in respect of $12 = 0$ and $34 = 0$.

It is evident that a similar pair of harmonicals (J) can be found at each of the 45 angles of the complete hexagon given in the conic $C = 0$, so that we have established the following

Theor. (II). *From each of the* 45 *intersections* p *of pairs of sides of the complete hexagon determined by six points in a conic, can be drawn two lines* (J) *synharmonicals in respect of the two intersecting sides, each line* (J) *containing two of the* 60 *points* (H) *of triple intersection (not Steiner's) of Pascal's lines; so that three of the* 90 *lines* (J) *pass through every point of the* 60 (H).

We have shewn that $J12.43$ passes through $H(12.43)\,56$: the other two lines (J) through that point are $J35.61$ and $J45.62$, as may be seen by writing the other two symbols for the point.

We see plainly that (ou) or $H(43.56)\,12$, and $(o_{,}u_{,})$ or $H(43.56)\,21$, are on $J43.56$; or, since $(ou_{,})$ and $(o_{,}u)$ are in $12 = 0$,

$$oo_{,} - \beta u_{,}u = 0 = 12.J43.56,$$

and, as above, $\quad oo_{,} - \lambda ee_{,} = 0 = 56.J12.43$:

whence follows, if we observe that $(eu_{,})$ and $(e_{,}u)$ are in $34 = 0$, as well as $(12.J12.43)$,

$$\beta u_{,}u - \lambda ee_{,} = 0 = 34.J56.21\,;$$

which proves that $J43.56$, $J12.43$, and $J56.21$ meet in a point which is not a point (H): and, because $J43.56$ is also $J56.43$, $J43.56$ $J12.56$ and $J43.21$ must equally meet in a point. Two such points (j) are thus found on each of the 90 lines (J), which gives us the following

Theor. (III). *The* 90 *lines* (J) *of the preceding Theorem go three together through* 60 *points* (j).

We may represent the intersection of $J43.56$, $J43.12$, and $J56.21$, by the symbol $j43^{56}_{12}$.

It may readily be shewn, that with the three equations

$$Aa + Bb = Cc,$$
$$Aa_{,} + Bb_{,} = Cc_{,},$$
$$Aa_{,,} + Bb_{,,} = Cc_{,,},$$

in which all the twelve A, a, &c. are linear functions of two variables, and in which C and C only vanishes with A and B, there is always given another system of three similar equations,

$$Da + Ea_{,} = Fa_{,,},$$
$$Db + Eb_{,} = Fb_{,,},$$
$$Dc + Ec_{,} = Fc_{,,},$$

in which F and F only vanishes with D and E. We have thus a simple analytical proof of the known properties, that if the nine lines (or points), (using point or line-coordinates),

$$\begin{matrix} a & b & c \\ a_{,} & b_{,} & c_{,} \\ a_{,,} & b_{,,} & c_{,,} \end{matrix}$$

form a triad of copolar (or coaxial) triangles when read horizontally, they form such a triad when read vertically. D, E, F are the axes (or poles) of the horizontal pairs, and A, B, C those of the vertical pairs.

Let now $34.61 = 0$ denote the equation to the point of intersection of the lines $34 = 0$ and $61 = 0$. The nine points

56.12,	12.34,	43.56,
23.45,	45.61,	23.61,
$H(56.21)\,34$,	$H(34.21)\,65$,	$H(34.65)\,12$,

form a coaxial triad; the common axis of each pair of triangles, read horizontally, being the pascalian 123456: for 12.45 is in directum with the pairs 56.12 and 12.34, 23.45 and 45.61, and with $H(56.21)\,34$ and $H(34.21)\,65$, on the pascalian 562134; 23.56 is in directum with a pair of the three H-points along 346512, and 34.61 with three pairs along 342165. The common axis X of a pair of triangles, read vertically, will have a point in directum with 56.12 and 23.45, 12.34 and 45.61, 43.56 and 23.61, which is none other than $G\,12.54.36$; a second point in directum with 56.12 and $H(56.21)\,34$, and with 12.34 and $H(34.21)\,65$; and this point can only be $j21^{34}_{56}$, the intersection of $J12.43$, $J12.65$, and $J34.65$; and a third point in directum with 23.45 and $H(56.21)\,34$, and with 45.61 and $H(34.21)\,65$, which third point must be $H(26.13)\,54$, the intersection of the pascalians 315462 and 261354. We have thus

a line X through $G\,12.54.36$, $j21^{34}_{56}$, and $H(26.13)\,54$,

and in like manner

a line X' through $G\,14.56.32$, $j41^{36}_{52}$, and $H(42.13)\,56$, and
a line X'' through $G\,16.52.34$, $j61^{32}_{54}$, and $H(64.13)\,52$.

We have then evidently

$$(145632)(532461)-\lambda(451326)(165234)=0=X.32,$$

X and 32 being the diagonals of the quadrilateral made by the four pascalians on the left, of which 145632 and 165234 meet in the G-point, and the other pair in the H-point in X. Further, in the same way,

$$(145632)(651342)-\lambda'(653124)(165234)=0=X'65;$$

by which and the preceding equation we obtain

$$\lambda'(532461)(653124)-\lambda(451326)(651342)=0=X_{,}.(513246),$$

for the three points (32.56), (24.51), (31.46), are in the pascalian 513246. The line $X_{,}$ must therefore contain the points (XX'), $H(26.13)\,54$, and $H(42.13)\,56$, two points both of X and of X'; or X, X', and $X_{,}$ are the same line.

In a similar way it can be proved that X and X'' are the same line, by the cyclical permutation of the digits 462 into 624.

We have found a line X which contains a G-point, three of the 60 H-points, and three of the 60 j-points; and we may designate it by the symbol X^{153}_{246}, in which the upper or the lower triplet may be cyclically permuted, or the triplets may be transposed, without changing the symbol. There will be one such line, and no more, passing through each of the 20 G-points, and consequently one also through each of the 60 j- and of the 60 H-points.

Take now the thrice three lines which intersect in the three points p, intersections of sides c of our complete hexagon, that are in the pascalian e, 126543; viz.

$$\begin{array}{lll} a123456, & a'652134, & a''435621, \\ b123546, & b'652314, & b''435261, \\ d126453, & d'654132, & d''431625. \end{array}$$

Since (aa') is $H(56.12)\,34$, and (dd') is $H(64.12)\,35$, we have, neglecting a constant,

$$ad'-a'd=0=eX^{123}_{456};$$

and since (aa'') is $H(43.21)\,65$, and (dd'') is $H(35.21)\,64$,

$$ad''-a''d=0=eX^{126}_{453};$$

whence, because $(a'a'')$ is $H(56.43)\,12$ and $(d'd'')$ is $H(25.43)\,16$,

$$d'a''-a'd''=0=eX^{314}_{625};$$

therefore X^{314}_{562}, X^{123}_{456}, and X^{126}_{453} meet in a point; that is, X^{123}_{456},

or X^{312}_{564}, or X^{231}_{645}, all the same line, meets in one point X^{126}_{453} and X^{314}_{562}; whence it follows, from the symmetry of the symbols, that it must meet X^{235}_{641} in the same point likewise. Thus the four lines

$$X^{123}_{456},\ X^{126}_{453},\ X^{143}_{256},\ X^{523}_{416},$$

all meet in a point, which may be denoted by any of the four symbols Φ^{123}_{546}, Φ^{523}_{146}, Φ^{143}_{526}, Φ^{126}_{543}; and the number of the points Φ must be fifteen.

We are now enabled to lay down the following theorems, which were discovered in the order in which they are written.

THEOR. IV. (A. Cayley, Esq.) *The sixty points* H *of Theor. I. lie three together on twenty lines* X.

THEOR. V. (Rev. Geo. Salmon). *Every line* X *of Theor. IV. contains a point* G *of Theor. I. and three points j of Theor. III.*

THEOR. VI. (Rev. Geo. Salmon). *The twenty lines* X *of Theor. IV. go four together through fifteen points* Φ.

Mr. Cayley's proof sent to me of Theor. IV. was not the same as the one above given; and Mr. Salmon's demonstration, communicated to me, of the first part of Theor. V. was different from that here adopted. I have made use of Mr. Salmon's discovery of the coaxial triad above exhibited to prove at once the two first of these three theorems. It is easily seen, as was first observed by the same gentleman, that there is another pair of triangles whose angles are H points and sides pascalians, having the same axis with the three triangles above employed. The proof given of Theor. VI. is Mr. Salmon's.

As the twenty points G of Steiner, each one the intersection of three of the sixty pascalian lines, lie four together on fifteen lines I, so do the twenty lines X discovered by Mr. Cayley, each containing three of the sixty points H, go four together through fifteen points Φ,—a remarkable correspondence.

We proceed next to the points in which the pascalians intersect by pairs; points which arrange themselves as in the following

THEOR. VII. *The* 60 *lines of Pascal intersect two and two in* 90 *points* m, 360 *points* r, 360 *points* t, 360 *points* z, *and* 90 *points* w. *A point* m *is determined by two hexagons having four common sides, no opposite pairs being the same in both hexagons: a point* r *is determined by any two hexagons having three common sides, of which two are contiguous: a point* t *is determined by any*

two hexagons having two common sides, which are not an opposite pair in both hexagons: a point z *is determined by any two hexagons having one common side; and a point* w *by any two which have no common side.*

Consider the three points m,

$$a\,123456,\quad a'\,231456,\quad a''\,312456,$$

$$m\,123654,\quad m'\,231654,\quad m''\,312654:$$

we have evidently

$$aa'a'' - mm'm'' = 0 = 54.\Theta = 54.56.M,$$

where M is a line containing the three points (m); and as we can obtain a different line M with every pair of the fifteen sides of the complete hexagon meeting on $C = 0$ (as 54 and 56 in the point 5), there must be 60 lines M. We may denote the line above found by the symbol $M456$, or $M654$. Thus we have

Theor. VIII. *The* 90 *points* m *of Theor. VII. lie three together on* 60 *lines* M.

It is, perhaps, worthy of observation, that the three pascalians, which determine on a fourth its three points m, meet in a point G; and the three that determine on a fourth its three points w, meet in a point H.

Take now the six points r, (ar), $(br_{,})$, &c.,

$$a\,123456,\quad b\,231456,\quad c\,312456,$$

$$r\,123645,\quad r_{,}\,231645,\quad r_{,,}\,312645,$$

$$a\,123456,\quad b\,231456,\quad c\,312456,$$

$$r'\,123564,\quad r''\,231564,\quad r'''\,312564.$$

Because $(ar'''r_{,})$ is $H(56.12)\,34$, $(r'r_{,,}b)$ is $H(64.12)\,35$, and (rcr'') is $H(45.12)\,36$, all in X^{123}_{456}, we have, observing the points p, $(ar_{,,})$, (br), $(cr_{,})$, &c.,

$$abc - rr_{,}r_{,,} \;= 0 = XR.45,$$

$$abc - r'r''r''' = 0 = XR'.56;$$

therefore $$rr_{,}r_{,,} - r'r''r''' = 0 = XR''.64.$$

Now the intersection of 45 and 56, or the point 5, is not on 64; it is therefore on the line R'', which contains the three r-points (rr'), $(r_{,}r'')$, and $(r_{,,}r''')$: and in like manner R, containing (ar), $(br_{,})$, and $(cr_{,,})$, passes through the point 6, and R', containing (ar'), (br''), and (cr'''), contains the point 4, on the conic C.

It is evident, from the consideration of the intersections of the line 64 with R and R', that R, R', and R'' meet in a point D. My first and hasty conclusion, as to the number

of these lines and points, was, that there were $120R$ going 20 together through the six points on the conic, and $40D$; but Mr. Salmon, by comparing the foregoing reasoning with a subsequent result of his own, that the points r lie *by twos* on certain lines passing through the angles of the given hexagon, first saw and first delivered the true enumeration, as stated in the following theorem; the authorship of which the reader will divide as he may think fit between Mr. Salmon and me:

THEOR. IX. *The* 360 *points* r *of Theor. VII. lie by sixes on* 60 *lines R, which pass by threes through twenty points D, and by tens through the six angles of the hexagon in the given conic.*

It needed but a glance at my own reasoning to reveal the truth about these numbers. Every line R passing through the point 6 will be exhibited by an equation like $X.R.45 = 0$, where 6 cannot appear in the third factor. Of such third factors there are ten, and consequently ten lines R through the point 6. The supposition that these ten are all, will bring before us every line of the twenty X six times; for X^{123}_{456} is also X^{456}_{123}, by which substitution can be obtained three equations containing the same X, $XR_{,}12 = 0$, $XR_{,}'23 = 0$, $XR_{,}''13 = 0$. To suppose that there are twenty lines R through the point 6, involves the existence of another equation $XR_{,,}45 = 0$ (as well as of $X'R_{,,,}45 = 0$ where X' is X^{132}_{456}): and it is easily seen above that no equation can be formed with the nine pascalians that meet X in its three H-points, of the form $XR_{,,}45 = 0$, except the one already formed $XR45 = 0$.

The line R above found may be denoted by the symbol $R123.6$, and contains, beside the three points (ar), $(br_{,})$, and $(cr_{,,})$, those made by the substitution in them of 5 for 4. The point D, in which meet $R123.4$, $R123.5$, and $R123.6$, is sufficiently defined by the symbol $D123$, where the triplet may be cyclically permuted, as may also the *triplet* in $R123.6$, &c.

Take next the four t-points

$$\begin{array}{llll} a\,123456, & a\,123456, & e\,213456, & e\,213456, \\ t\,126435, & t'\,125463, & t_{,}\,216435, & t_{,}'\,215463: \end{array}$$

From
$$at_{,} - et = 0 = 12T,$$
$$t_{,}'a - et' = 0 = 12T',$$
comes
$$t't_{,} - tt_{,}' = 0 = 12T''.$$

The line T'', containing the two t-points, (tt') and $(t_{,}t_{,}')$,

meets in the same point E the two lines T and T', each containing two points t. If we take the four t-points

$$a\,432165,\quad a\,432165,\quad a\,342165,\quad a\,342165,$$

$$t\,435126,\quad t_2\,436152,\quad t_3\,345126,\quad t_4\,346152,$$

we obtain three similar lines $T_, T'_, T''_,$, meeting in $E_,$, of which the first, $T_,$, contains (at), as does $\dot{T}$. Two such lines T may be shewn to pass through any t-point, whence the

THEOR. X. *The* 360 *points* t *of Theor. VII. lie in pairs on* 360 *lines* T, *which pass three together through* 120 *points* E.

The six lines $T, \underset{.}{T'}, T'', \underset{.}{T_{,}}, \underset{.}{T'_{,}}, \underset{.}{T''_{,}}$ may be in order defined by the symbols $T345\dot{6}$, $T546\dot{3}$, $T643\dot{5}$, $T216\dot{5}$, $T615\dot{2}$, $T512\dot{6}$. If it were not for their triple intersections in the points E, the lines T would of course be unworthy of notice; and the reader may still probably pronounce them unfit for the company into which they are thrust.

For linear relations among the points z and w of Theor. VII. I have searched in vain.

§. II.—*Secondary and non-linear relations among the lines and points about the Hexagram.*

The first place is due to a theorem of Mr. Salmon's, on account of a curious analogy which it exhibits between the 60 points j and the 60 pascalians.

The point G12.34.56 is the pole of the following six triangles, whose angular points are the nine H-points, and the nine p-points on the three pascalians which meet in it.

	123456,	143652,	163254;
(A)	J25.63,	J45.61,	J12.34,
(B)	J63.41,	J23.45,	J56.12,
(C)	J25.41,	J23.61,	J56.34,
(D)	36,	45,	12,
(E)	52,	61,	34,
(F)	41,	23,	56;

by which is to be understood that in the triangle J25.63, J45.61, J12.34, the angle opposite the side J25.63 is on the line 123456, and so on.

The axes of the three triangles (D), (E), (F) are the three pascalians 123654, 143256, 163452, which meet in the point G12.36.54, a property which was first pointed out, I believe, by Mr. Cayley in *Crelle's Journal.*

The axis of the pair (A), (D) is 163452, that of the pair $(B)(D)$ is 123654, wherefore the axis of the pair $(A)(B)$ passes through G12.36.54; and by considering in the same way the three $(B)(C)(F)$, we see that the axis of the pair $(B)(C)$ passes through the same point; whence it follows that the axis of the pair $(A)(C)$ goes also through G12.36.54, which may be called the *parapole* of the copolar triad $(A)(B)(C)$, and likewise of the triad $(D)(E)(F)$.

The points of the three axes in which meet the homologous sides of (A) (B) (C) are all j-points, as is evident by inspection; whence we deduce the

THEOR. XI. (Rev. Geo. Salmon). *The* 60 *points* j *of Theor. III. lie in threes on* 60 *lines* L, *which meet by threes in the* 20 *points G of Steiner.*

Six copolar triangles will have their 15 axes in general passing three together through 20 points; but it is curious that the whole of the 20 points in the copolar hexad before us reduce themselves to G12.36.54, the common *parapole*, as G12.34.56 is the common pole, of all the twenty triads. The properties of the points j, that they lie in threes on twenty lines X passing four together through fifteen points Φ, and also in threes on sixty lines L passing three together through twenty points G, have so striking a resemblance to those of the pascalian lines, which go in threes through twenty points G lying four and four on fifteen lines I, and also in threes through sixty points H, lying three and three on twenty lines X, that the suspicion naturally arises of some reciprocal relation, as that the points j might be the polars of the pascalians with respect to the conic C. Such a relation has however not yet been discovered.

Consider again the four H-points

$$u\,125643,\quad e_{,}\,126543,\quad u_{,}\,561243,\quad c\,651243,$$

$$a\,216543,\quad o_{,}\,215643,\quad a_{,}\,652143,\quad o\,562143.$$

Neglecting a constant, we have

$$\left.\begin{aligned} ao_{,} - ue_{,} &= 0 = K\,12 \\ a_{,}o - u_{,}e &= 0 = K'12 \\ ae_{,} - o_{,}u &= 0 = K\,56 \\ a_{,}e - ou_{,} &= 0 = K'56 \end{aligned}\right\}\ldots\ldots\ldots.(A),$$

where the line K contains (au) and $(o_{,}e_{,})$, while K' contains $(a_{,}u_{,})$ and (oe). The intersection of K with K' is on each of

the four loci

$$aoa_,o_, - ueu_,e_, = 0 = 12.34F,$$
$$aeo_,u_, - a_,e_,ou = 0 = 12F_,I,$$
$$aea_,e_, - ouo_,u_, = 0 = 12.34F_,,$$
$$aoe_,u_, - a_,o_,eu = 0 = 56FI;$$

for the first and third of these contain each four points of 34, and the second and fourth each four points of I12.34.56. The point (KK') is on neither of the conics F, $F_,$; for both contain (au) and $(o_,e_,)$ two points of K: and since (12.56) is in directum with (au) and $(a_,u_,)$ along J12.56, (KK') must be the intersection of I with 34. The point K containing H.(12.56) 43 and H(12.65) 43 is identified by the symbol $K(12.\overline{56})\,43$, which is the same with $K(12.\overline{65})\,43$. There will manifestly pass through H(12.56) 43 two other such lines, $K(64.\overline{31})\,25$ and $K(54.\overline{32})\,61$, so that the number of lines must be equal to that of the lines J, and each K will meet a line I on one of the 15 chords of the conic C; which properties are expressed as follows:

THEOR. XII. *Through each of the* 60 *points* H *of Theor. I. go three lines* K, *each of which contains a second point* H, *and passes through the intersection* (i) *of one of Steiner's* 15 *lines* I *with one of the sides* (c) *of Pascal's complete hexagon: the number of the lines* K *is* 90, *and that of the points* (i) *is* 45, *three of which lie on each of the lines* I, c.

The equations (A) give

$$a^2 - u^2 = 0 = K(12.\overline{56})\,43.J12.56,$$
$$o_,^2 - e_,^2 = 0 = K(12.\overline{56})\,43.J12.65\text{; whence comes}$$

THEOR. XIII. (Rev. G. Salmon). *At every point* H *of Theor. I., the nine lines* (A) (J) (K) *of Theor. I., III., XII., form three harmonic pencils.*

By the equations (see the preceding page)

$$au_, - e_,o = 0 = 12\,K(12.\overline{34})\,56,$$
$$oa_, - eu_, = 0 = 12\,K(12.\overline{65})\,34,$$
$$aa_, - ee_, = 0 = 12\;J\,(34.56),$$

we see that J(34.56), $K(12.\overline{34})\,56$, and $K(12.\overline{65})\,34$ meet in a point, and in the same line J we must also have the point [J(43.65), $K(12.\overline{34})\,65$, $K(12.\overline{65})\,43$], whence follows

THEOR. XIV. *In every one of the* 90 *lines* J *of Theor. III. are two points, in each of which meet a pair of the lines* K *of Theor. XII.*

The conic sections F and $F_{,}$, above found, deserve to be noticed. $F = 0$ contains the eight points

$$au,\quad ae,\quad ou,\quad oe,$$
$$a_{,}u_{,},\quad a_{,}e_{,},\quad o_{,}u_{,},\quad o_{,}e_{,},$$

or $H(12.56)43$, $H(34.56)12$, $H(43.56)21$, $H(65.12)43$,
$H(12.56)34$, $H(34.56)21$, $H(43.56)12$, $H(65.12)34$,

and $F_{,}$ contains the first and fourth pairs of these besides four other H-points. A third conic $F_{,,}$, given in the equation

$$aue_{,}o_{,} - a_{,}u_{,}eo = 0 = 34F_{,,}I,$$

meets F in the second and third of the above pairs, containing also the four other H-points (ao), $(a_{,}o_{,})$, $(e_{,}u_{,})$, and (eu), which are upon $F_{,}$. We can find three such conics, each containing eight H-points, in connexion with any of the fifteen lines I, and thus prove the

THEOR. XV. *Through every point of the* 60 H *of Theor. I. go six conics* F, *each one containing eight points* H, *the number of the conics* F *being forty-five.*

By considering the triangle formed by the G-points on three pascalians meeting in a point H, which is copolar with a triangle formed by three points p, we obtain the interesting

THEOR. XVI. (Rev. Geo. Salmon). *Through each of Steiner's* 20 *points* G *go three lines each containing three points* i *of Theor. XII.*

THEOR. XVII. (Rev. Geo. Salmon). *The sixty pascalian lines form twenty hexads of copolar triangles, the poles being the twenty points* D *of Theor. IX.*

From this last theorem it is easily deduced that the four lines $R123.4$, $M465$, $M456$, 56 meet in a point, whence follows

THEOR. XVIII. (Rev. Geo. Salmon). *There are sixty points in each of which a side of the complete hexagon meets two lines* M *of Theor. VIII., and a line* R *of Theor. IX.*

THEOR. XIX. (Rev. Geo. Salmon). *The* 360 *points in which the* 60 *pascalians meet the lines connecting non-opposite vertices of their defining hexagons lie by threes on* 120 *lines.*

It is easily shewn, by considering the pascalians

$$b\,123456,\quad d\,213456,\quad b_{,}123456,\quad d_{,}213456,$$
$$c\,123564,\quad f\,213564,\quad c_{,}124536,\quad f_{,}214536,$$
$$g\,123645,\quad k\,213645,\quad g_{,}125346,\quad k_{,}215346,$$

that the following equations are true,

$$bf - dc = 0 = 12.R123.4,$$
$$bk - dg = 0 = 12.R123.6,$$
$$bf_{,} - dc_{,} = 0 = 12.R612.3,$$
$$bk_{,} - dg_{,} = 0 = 12.R612.5;$$

whence are obtained

$$fg_{,} - ck_{,} = 0 = 12.J53.46,$$
$$fc_{,} - f_{,}c = 0 = 12.T356\dot{4},$$
$$kc_{,} - gf_{,} = 0 = 12\,I21.36.45;$$

which shew that $R123.4$, $R612.5$, and $J53.46$ meet in a point; as do $R123.4$, $R612.3$, and $T356\dot{4}$, and also $R123.6$, $R612.3$, and $I21.36.45$.

In a similar way, from the pascalians

$$b\,123456,\quad d\,213456,\quad b\,123456,\quad d\,213456,$$
$$s\,126435,\quad r\,216435,\quad s_{,}\,124653,\quad r_{,}214653,$$
$$t\,125463,\quad v\,215463,\quad t_{,}\,126354,\quad v_{,}\,216354,$$

are deduced

$$br - ds = 0 = 12\,T345\dot{6},$$
$$br_{,} - ds_{,} = 0 = 12\,T654\dot{3},$$
$$bv - dt = 0 = 12\,T546\dot{3},$$
$$bv_{,} - dt_{,} = 0 = 12\,T453\dot{6};$$

whence come

$$rs_{,} - r_{,}s = 0 = 12\,J64.35,$$
$$rt_{,} - v_{,}s = 0 = 12\,R612.4,$$
$$vt_{,} - v_{,}t = 0 = 12\,I12.63.54,$$

by which we see that $T345\dot{6}$, $T654\dot{3}$, and $J64.35$ meet in a point; as do also $T345\dot{6}$, $T453\dot{6}$, and $R612.4$, and also $T546\dot{3}$, $T453\dot{6}$, and $I12.63.54$.

But $J53.46$ is $J35.64$, and contains therefore the intersection of $R125.6$ and $R412.3$, as well as that of $R123.4$ and $R612.5$; it passes also through the intersection of $T563\dot{4}$ and $T436\dot{5}$, as well as of $T345\dot{6}$ and $T654\dot{3}$. The line $T356\dot{4}$ has no second symbol, so that there are as many points $(TRR_{,})$ as lines T, *i.e.* 360; the line $R612.4$ will contain the intersection of $T543\dot{6}$ and $T435\dot{6}$ as well as that of $T345\dot{6}$ and $T453\dot{6}$, and since $R126.4$ and $R261.4$ are the same with $R612.4$, there will be two other points $(RTT_{,})$ given with each of these forms. The line $I21.36.45$, is $I21.36.54$, or $I21.63.45$, or $I21.63.54$, or $I36.21.45$, &c.,

giving twelve forms, each of which determines a different point $(ITT_{,})$; but there are in the same line I only six points $(IRR_{,})$, because the point $(R123.6, R612.3)$ is not changed by putting 1 for 2.

We may now add the following theorems:

THEOR. XX. *In every one of Steiner's* 15 *lines* I *are six points of intersection of pairs of lines* R *of Theor. IX., and twelve points of intersection of pairs* T *of Theor. X.*

THEOR. XXI. *In each of the* 90 *lines* J *of Theor. III. are two intersections of pairs* R *of Theor. IX. and two of pairs* T *of Theor. X.*

The position of the points $(RTT_{,})$ is expressed in

THEOR. XXII. *The* 360 *lines* T *of Theor. X. form twenty hexads of copolar triangles, whose poles are the twenty points* D *of Theor. IX.*

THEOR. XXIII. *In each of the* 360 *lines* T *is an intersection of a pair of lines* R.

If Z^{4653}_{5436} denote the line containing the two z-points of Theor. VII., $\left\{\begin{matrix}124653\\125436\end{matrix}\right\}$ and $\left\{\begin{matrix}214653\\215436\end{matrix}\right\}$, it is easy to prove the following propositions—

There are 360 points of the form $(R123.4.R345.6.Z^{4653}_{5436})$,

There are 360 points of the form $\{K(45.\overline{12}.)63.T356\dot{4}.Z^{3564}_{6345}\}$,

There are 360 points of the form $(R345.6\ T654\dot{3}\ Z^{6345}_{4653})$.

THEOR. XXIV. *In each of the* 90 *lines* K *of Theor. XII. are two intersections of pairs of lines* R *of Theor. IX.*

The 60 lines R have remarkable properties; each of these containing 6 points r, a point $(RR_{,}R_{,,})$, a point $(RMM_{,}c)$, three points $(RR_{,}I)$, six points $(RR_{,}J)$, six points $(RR_{,}K)$, six points $(RTT_{,})$, six points (RTZ), twelve points $(RR'T)$, and twelve points $(RR_{,}Z)$, besides one of the six given points on the conic C.

THEOR. XXV. *In each of the* 180 *harmonicals* h, *conjugate to Pascal's* 60 *lines in respect of opposite sides of their defining hexagons, are two points* q, *at each of which it is synharmonic with another harmonical* h *in respect of a pascalian line and a line* c *joining opposite vertices of its defining hexagon, the points* q *being in number* 180; *also four points* k, *each one the intersection of a second line* h *with a pascalian line; and four points* l, *each one the intersection of a second line* h *with a line* N *connecting an* H*-point (Theor. I.) with one of the six given points on the conic; and the number of the lines* N *is* 360, *equal to that of the points* l.

By considering the pascalians that meet upon one of Steiner's 15 lines I, it is easy to prove the two first of the following theorems:

THEOR. XXVI. *There are* 60 *conics* (e) *each containing six of the* 45 *intersections* p *of the sides of the complete hexagon.*

THEOR. XXVII. *There are* 30 *curves* (f) *of the third degree each containing* 12 *points* z *of Theor. VII. and* 120 *conics* (d), *each containing* 6 *points* z.

THEOR. XXVIII. *There are* 60 *conics* T, *each containing six points* t *of Theor. VII.*

THEOR. XXIX. (Rev. Geo. Salmon). *There are* 120 *conics* S, *each containing six points* t *of Theor. VII., and one of the six given points of the conic* C.

It is easy to find thousands of conics, each fulfilling six or more conditions at the intersections of the leading lines of the figure; and the curves of higher degrees, which fulfil remarkable conditions in the figure, amount to an enormous number.

It may be worth while to shew how the figure arranges itself when the given conic is reduced to a pair of lines.

THEOR. XXX. *If in each of two intersecting right lines* B *and* C *three points* P *be taken, and the six points* P *be connected by nine lines* c, *meeting in* 18 *points* p; *there are given two points* (*Steiner's*) G, *through each of which pass three pascalian lines* A, *every* A *meeting* 6 *lines* c *in points* p, *and three* c *in points* q.

THEOR. XXXI. *The line* J, *synharmonic at any point* p *with the line* A *through it in respect of two lines* c, *passes through two points* q, *at each of which it is synharmonic with another line* J, *in respect of the pascalian line* A *and the line* c, *which meet thereat.*

THEOR. XXXII. *The* 36 *points* pq *lie* 8 *together,* 4p *and* 4q, *on* 18 *conics F.*

We may consider the lines B and C of Theor. XXX. to be the section of a given ruled hyperboloid made by any tangent plane, the six points P being six given generating lines, of either kind three, meeting the plane: the nine lines c are now given tangent planes; the six lines A are given pascalian planes, intersecting three together in two given (Steiner's) lines G, the existence of which last Plücker has shewn in his Solid Geometry: the harmonic pencils p and q are now given bunches of harmonic planes, viz. eighteen bunches $(AJcc_{,})$, and eighteen bunches $(AJJ_{,}c)$. I am not aware that these have been before noticed.

It is unnecessary to observe that all the properties above given have corresponding reciprocal polars.

The greatest part of the foregoing theorems were published in the *Manchester Courier* in June and August last.

Croft Rectory, near Warrington, Dec. 3, 1849.

NOTES ON ELLIPTIC FUNCTIONS (FROM JACOBI).

Translated by ARTHUR CAYLEY.

THE following I believe not very generally known theorems in elliptic functions are mere translations from Jacobi's "Note sur une nouvelle application de l'analyse des fonctions elliptiques à l'algèbre," (*Crelle*, tom. VII. p. 41), and from the addition to the notice by him of the third supplement to Legendre's "Theorie des fonctions elliptiques" (*Crelle*, t. VIII. p. 412).

THEOREM. Let

$$R = z(z-1)\left(z - \frac{1}{1-k^{-1}p^2}\right)\left(z - \frac{1}{1-kp^2}\right),$$

$$= z^4 - az^3 + bz^2 - cz;$$

$$\begin{cases} a = \dfrac{3-2ap^2+p^4}{1-ap^2+p^4}, \quad b = \dfrac{3-ap^2}{1-ap^2+p^4}, \quad c = \dfrac{1}{1-ap^2+p^4}, \\ a = k + \dfrac{1}{k}, \end{cases}$$

$$R = z^2 - \tfrac{1}{2}az + i_1 + \cfrac{1}{M_1z+m_1 + \cfrac{1}{M_2z+m_2 + \ddots \cfrac{1}{M_{n-1}z+m_{n-1} + \cfrac{1}{\cfrac{\sqrt{R}+z^2-\frac{1}{2}az+i_n}{q_n(r_nz-1)}}}}},$$

where if $p = p_1 = \sqrt{k}\sin am \dfrac{u}{\sqrt{k}}$, $p_n = \sqrt{k}\sin am \dfrac{nu}{\sqrt{k}}$,

$$i_n = \frac{1}{2(1-ap^2+p^4)}\left(1 - \frac{p^2}{p_{2n}^2}\right),$$

$$r_n = \left(1 - \frac{p^2}{p_{2n+1}^2}\right),$$

odd $$q_n = \frac{1}{4(1-ap^2+p^4)^2}\left\{\frac{\left(1-\frac{p^2}{p_2^2}\right)\left(1-\frac{p^2}{p_6^2}\right)\cdots\left(1-\frac{p^2}{p_{2n}^2}\right)}{\left(1-\frac{p^2}{p_4^2}\right)\left(1-\frac{p^2}{p_8^2}\right)\cdots\left(1-\frac{p^2}{p_{2n-2}}\right)}\right\}^2,$$

even $$q_n = -\left\{\frac{\left(1-\frac{p^2}{p_4^2}\right)\left(1-\frac{p^2}{p_8^2}\right)\cdots\left(1-\frac{p^2}{p_{2n}^2}\right)}{\left(1-\frac{p^2}{p_2^2}\right)\left(1-\frac{p^2}{p_6^2}\right)\cdots\left(1-\frac{p^2}{p_{2n-2}^2}\right)}\right\}^2.$$

So that in general $\quad q_n q_{n-1} = -\dot{\iota}_n^2.$

The preceding forms may also be written

$$\dot{\iota}_n = \frac{1}{2(1-ap^2+p^4)} \frac{p_{2n+1}p_{2n-1}}{p^2_{2n}} (1-p^2p^2_{2n}),$$

$$r_n = \frac{p_{2n+2}p_{2n}}{p^2_{2n+1}} (1-p^2p^2_{2n});$$

and n odd,

$$q_n = \frac{p^2p^2_{2n+1}}{4(1-ap^2+p^4)} \left(\frac{p_4p_8\cdots p_{2n-2}}{p_2p_6\cdots p_{2n}}\right)^4 \frac{(1-p^2p_2^2)(1-p^2p_6^2)\dots(1-p^2p^2_{2n})}{(1-p^2p_4^2)(1-p^2p_8^2)\dots(1-p^2p^2_{2n-2})};$$

n even,

$$q_n = \frac{p^2_{2n+1}}{p^2} \left(\frac{p_2p_6\cdots p_{2n-2}}{p_4p_8\cdots p_{2n}}\right)^4 \frac{(1-p^2p_4^2)(1-p^2p_8^2)\dots(1-p^2p^2_{2n})}{(1-p^2p_2^2)(1-p^2p_6^2)\dots(1-p^2p^2_{2n-2})}.$$

When R is of a higher degree than the fourth, the continued fraction into which $\sqrt{R}$ can be converted depends on the formulæ for the multiplication of the Abelian transcendants.

The theorem at the end of Legendre's third supplement may be extended to the more general integral

$$\int_0^x \frac{dx}{\sqrt{\{x(1-x)(1-\kappa\lambda x)(1+\kappa x)(1+\lambda x)\}}},$$

which in the case $\lambda = 1$ coincides with Legendre's integral, and which is likewise always reducible to the sum of two elliptic functions F, having the same amplitude, but the moduli of which are not in general (as they are in Legendre's integral) complementary; but on the contrary (by assigning proper values to κ, λ).may be any quantities whatever. In fact if b and c be any two moduli, b' and c' the complementary moduli $\sqrt{(1-b^2)}$, $\sqrt{(1-c^2)}$, and

$$\kappa = \left(\frac{c'-b'}{b-c}\right)^2, \quad \lambda = \left(\frac{c'-b'}{b+c}\right)^2,$$

or

$$b = \frac{\sqrt{\kappa}+\sqrt{\lambda}}{\sqrt{(1+\kappa)}\sqrt{(1+\lambda)}}, \quad c = \frac{\sqrt{\kappa}-\sqrt{\lambda}}{\sqrt{(1+\kappa)}\sqrt{(1+\lambda)}},$$

$$b' = \frac{1-\sqrt{(\kappa\lambda)}}{\sqrt{(1+\kappa)}\sqrt{(1+\lambda)}}, \quad c' = \frac{1+\sqrt{\kappa\lambda}}{\sqrt{(1+\kappa)}\sqrt{(1+\lambda)}}.$$

The substitution

$$\sqrt{x} = \frac{b'+c'}{\sqrt{(1-b^2\sin^2\phi)}+\sqrt{(1-c^2\sin^2\phi)}}$$

gives

$$\int_0^x \frac{dx}{\sqrt{\{x(1-x)(1-\kappa\lambda x)(1+\kappa x)(1+\lambda x)\}}} = \frac{b'+c'}{2}[F(b,\phi)+F(c,\phi)]$$

$$\int_0^x \frac{\sqrt{x}\,dx}{\sqrt{\{(1-x)(1-\kappa\lambda x)(1+\kappa x)(1+\lambda x)\}}} = \frac{(b'+c')^2}{2(c'-b')}[F(b,\phi)-F(c,\phi)],$$

where $$\frac{(b'+c')^2}{c'-b'} = \frac{1}{\sqrt{(\kappa\lambda)}\sqrt{(1+\kappa)}\sqrt{(1+\lambda)}};$$

and where also

$$\sin^2\phi = \frac{(1+\kappa)(1+\lambda)x}{(1+\kappa x)(1+\lambda x)}, \quad \cos^2\phi = \frac{(1-x)(1-\kappa\lambda x)}{(1+\kappa x)(1+\lambda x)},$$

$$1-b^2\sin^2\phi = \frac{\{1-\sqrt{(\kappa\lambda)}\,x\}^2}{(1+\kappa x)(1+\lambda x)}, \quad 1-c^2\sin^2\phi = \frac{(1+\sqrt{\kappa\lambda}\,x)^2}{(1+\kappa x)(1+\lambda x)};$$

which lead without difficulty to the preceding results. Also the limits of ϕ are 0 and $\frac{1}{2}\pi$ when those of x are 1 and 0.

Thus in general the sum and the difference of two elliptic functions F with the same amplitude and with arbitrary moduli possess the properties of the first class of Abelian integrals, or those in which the function under the root rises to the fifth or sixth order. This remark, which was first (in the case of complementary moduli) made by Legendre, and which, as the preceding investigation shews, is applicable to any two moduli, is important in the theory of elliptic functions, and may besides be of assistance in the theory of Abelian integrals.

If in the preceding formulæ λ is supposed negative, there is a pair of imaginary moduli. Thus if $b^2 = e+fi$, $c^2 = e-fi$, then if $-\lambda$ be written instead of λ,

$$\kappa = \frac{\sqrt{\{(1-e)^2+f^2\}}+e-1}{\sqrt{(e^2+f^2)}-e}, \quad \lambda = \frac{\sqrt{\{(1-e)^2+f^2\}}+e-1}{\sqrt{(e^2+f^2)}+e}.$$

The addition of the two results gives

$$\int_0^\phi \frac{d\phi}{\sqrt{\{1-(e+fi)\sin^2\phi\}}}$$

$$= \frac{1}{\sqrt{\{\sqrt{(1-e)^2-f^2}+e+1\}}}\int_0^x \frac{dx}{\sqrt{\{x(1-x)(1+\kappa\lambda x)(1+\kappa x)(1-\lambda x)\}}}$$

$$+\frac{i\sqrt{\{\sqrt{(1-e)^2+f^2}+e-1\}}}{\sqrt{\{(1-e)^2+f^2\}}-e+1}\int_0^x \frac{dx\sqrt{x}}{\sqrt{\{(1-x)(1+\kappa\lambda x)(1+\kappa x)(1-\lambda x)\}}}.$$

There are an indefinite number of cases in which imaginary moduli may be reduced to real ones. For in the case of a transformation of the n^{th} order a modulus may be transformed into as many other moduli as there are units in the sum of the factors of n; and of these, if the original modulus be real, only so many are real as there are factors of n, all the others are imaginary moduli transformable into the same real modulus. Thus the integral

$$\int_0^x \frac{dx\, x^{\pm\frac{1}{2}}}{\sqrt{\{(1-x)(1+\kappa\lambda x)(1+\kappa x)(1-\lambda x)\}}},$$

where κ and λ are positive, may in an indefinite number of cases be transformed into elliptic integrals with real moduli. And on the other hand the equation last found is probably the simplest representation of an elliptic integral F with an imaginary modulus in the form $P + Qi$; and so the theory of elliptic integrals with imaginary moduli conducts of necessity to the first class of Abelian integrals.

ON THE TRANSFORMATION OF AN ELLIPTIC INTEGRAL.

By Arthur Cayley.

The following is a demonstration of a formula proved incidentally by Mr. Boole (*Journal*, vol. II. p. 7, New Series), in a paper "On the Attraction of a Solid of Revolution on an External Point."

Let $$U = \int_{-1}^{1} \frac{dx}{[(1-x^2)\{1-(mx+n)^2\}]^{\frac{1}{2}}}.$$

Then, assuming $$ix = \frac{a+iy}{1-iay},$$

(so that $x = \pm 1$ gives $y = \pm 1$), we obtain

$$1 - x^2 = \frac{(1+a^2)(1-y^2)}{(1-iay)^2},$$

$$mx + n = \frac{(n-ima)+(m-ina)\,y}{1-iay}.$$

Assume therefore

$$ia + (n-ima)(m-ina) = 0,$$

whence $$-ia = \frac{(1-m^2-n^2)+\Delta}{2mn},$$

where $$\Delta^2 = 1 + m^4 + n^4 - 2m^2 - 2n^2 - 2m^2n^2,$$

$$1-(mx+n)^2 = \frac{1-(n-ima)^2}{(1-iay)^2}\{1-(m-ina)^2y^2\},$$

$$dx = \frac{(1+a^2)\,dy}{(1-iay)^2},$$

$$U = \sqrt{\left\{\frac{1+a^2}{1-(n-ima)^2}\right\}}\int_{-1}^{1}\frac{dy}{[(1-y^2)\{1-(m-ina)^2y^2\}]^{\frac{1}{2}}},$$

i.e. $$U = 2\sqrt{\left\{\frac{1+a^2}{1-(n-ima)^2}\right\}}\int_{0}^{1}\frac{dy}{[(1-y^2)\{1-(m-ina)^2y^2\}]^{\frac{1}{2}}}.$$

Or, since $$n - ima = \frac{1-m^2+n^2+\Delta}{2n},$$

$$m - ina = \frac{1+m^2-n^2+\Delta}{2m},$$

we have

$$1-(n-ima)^2 = -\frac{\Delta}{2n^2}(\Delta+1-m^2+n^2),$$

$$1+a^2 = -\frac{\Delta}{2m^2n^2}(\Delta+1-m^2-n^2);$$

and therefore

$$\frac{1+a^2}{1-(n-ima)^2} = \frac{1}{m^2}\,\frac{\Delta+1-m^2-n^2}{\Delta+1-m^2+n^2}$$

$$= \frac{1}{m^2}\,\frac{(1-m^2-n^2+\Delta)(1-m^2+n^2-\Delta)}{(1-m^2+n^2+\Delta)(1-m^2+n^2-\Delta)} = \frac{2(1+m^2-n^2+\Delta)}{4m^2},$$

$$U = \frac{1}{m}\sqrt{\{2(1+m^2-n^2+\Delta)\}}\int_0^1\frac{dy}{\sqrt{\left[(1-y^2)\left\{1-\left(\frac{1+m^2-n^2+\Delta}{2m}\right)^2y^2\right\}\right]}}.$$

Write $$k = \frac{1+m^2-n^2+\Delta}{2m},\quad \lambda^2 = \frac{4m}{(1+m)^2-n^2};$$

then $$U = \frac{4\sqrt{k}}{\lambda}\,\frac{1}{\sqrt{\{(1+m)^2-n^2\}}}\int_0^1\frac{dy}{\sqrt{\{(1-y^2)(1-k^2y^2)\}}}.$$

Also λ and k are connected by the relation that exists for the transformation of the second order, viz.

$$\lambda = \frac{2\sqrt{k}}{1+k},$$

as may be immediately verified; hence, assuming

$$y = \frac{\lambda z}{\sqrt{k}} \sqrt{\left(\frac{1-z^2}{1-\lambda^2 z^2}\right)},$$

which gives

$$\int_0^1 \frac{dy}{\sqrt{\{(1-y^2)(1-k^2y^2)\}}} = \frac{\lambda}{\sqrt{k}} \int_0^1 \frac{dz}{\sqrt{\{(1-z^2)(1-\lambda^2 z^2)\}}},$$

$$U = \frac{4}{\sqrt{\{(1+m)^2 - n^2\}}} \int_0^1 \frac{dz}{\sqrt{\left\{(1-z^2)\left(1 - \frac{4m}{(1+m)^2 - n^2} z^2\right)\right\}}};$$

that is

$$\int_{-1}^1 \frac{dx}{\sqrt{[(1-x^2)\{1-(mx+n)\}^2]}} = 4\int_0^1 \frac{dz}{\sqrt{(1-z^2)[\{(1+m)^2-n^2\}-4mz^2]}}.$$

Write $x = \cos\theta$, $z = \cos\frac{1}{2}\phi$,

$$\int_0^\pi \frac{d\theta}{\sqrt{\{1-(m\cos\theta+n)^2\}}} = \int_0^\pi \frac{d\phi}{\sqrt{(1+m^2-n^2-2m\cos\phi)}}.$$

Or if $m = \frac{r}{a}$, $n = -\frac{iz}{a}$,

$$\int_0^\pi \frac{d\theta}{\sqrt{\{a^2+(z+ir\cos\theta)^2\}}} = \int_0^\pi \frac{d\phi}{\sqrt{(a^2+r^2+z^2-2ar\cos\phi)}},$$

the formula in question.

ON GENERAL DIFFERENTIATION.

By the Rev. W. Center.

PART III.

It is the object of the present paper to investigate the general differential coefficients of the transcendental functions $\log x$, $\sin ax$, and $\cos ax$; and to exhibit verifications of the resulting forms. In the prosecution of this design, the important subject of the complementary function has not been overlooked; a subject so essential to the proper interpretation

of our results. In the investigation of the general differential coefficient of $\log x$, the symbolical value of $\log x$, given in the preceding paper, appears of considerable importance.

If we take the integral $\int_0^\infty e^{-ax} da = x^{-1}$, and integrate both sides relatively to x, we have

$$-\int_0^\infty e^{-ax} a^{-1} da = \log x + C = (-\lceil 0.x^0 + C_0') + C \ldots .(1).$$

Supposing θ a positive proper fraction, let us now operate on (1) by $\left(\frac{d}{dx}\right)^\theta$ according to the *formulæ* already deduced, when we have simultaneously

$$(-1)^{\theta+1} \int_0^\infty e^{-ax} a^{\theta-1} da = \left(\frac{d}{dx}\right)^\theta \log x + \left(\frac{d}{dx}\right)^\theta C$$

$$= (-1)^{\theta+1} \frac{\lceil\theta}{x^\theta} + \left(\frac{d}{dx}\right)^\theta (C_0' + C).$$

But by the properties of C_0', already explained, we have

$$\left(\frac{d}{dx}\right)^\theta C_0' = \left(\frac{d}{dx}\right)^\theta C_0' x^0 = C_0' \left(\frac{d}{dx}\right)^\theta x^0 = C_0'.0 = 0\,;$$

also $$\left(\frac{d}{dx}\right)^\theta C = 0.$$

Hence $$\left(\frac{d}{dx}\right)^\theta \log x = (-1)^{\theta+1} \int_0^\infty e^{-ax} a^{\theta-1} da = (-1)^{\theta+1} \frac{\lceil\theta}{x^\theta} \ldots (2).$$

This important result, embodying the recognized integral

$$\int_0^\infty e^{-ax} a^{\theta-1} da = \frac{\lceil\theta}{x^\theta},$$

and giving an extension to its meaning in its new connexion with the function $\log x$, may be regarded as the fundamental form, from which all others of the same nature may be derived.

Let us now operate on (2) by $\left(\frac{d}{dx}\right)^{-n}$, n being a positive integer, and we have simultaneously

$$\left(\frac{d}{dx}\right)^{-n+\theta} \log x = (-1)^{\theta+1} \left(\frac{d}{dx}\right)^{-n} \int_0^\infty e^{-ax} a^{\theta-1} da + \left(\frac{d}{dx}\right)^{-n} 0,$$

$$= (-1)^{\theta+1} \lceil\theta \left(\frac{d}{dx}\right)^{-n} \frac{1}{x^\theta} + \left(\frac{d}{dx}\right)^{-n} 0\,;$$

or $\left(\frac{d}{dx}\right)^{-n+\theta}\log x = (-1)^{n+\theta+1}\int_0^\infty e^{-ax}a^{\theta-n-1}da$ + compl. function,

$$= (-1)^{n+\theta+1}\,\Gamma\overline{-n+\theta}.x^{n-\theta} + C_{n-1}x^{n-1} + \ldots + C_1x + C_0 \ldots (3).$$

Also, since $(1-\theta)$ is a positive proper fraction, we may in (3) write $(1-\theta)$ for θ, and $n+1$ for n; whence

$$\left(\frac{d}{dx}\right)^{-n-\theta}\log x = (-1)^{n-\theta+1}\,\Gamma\overline{-n-\theta}.x^{n+\theta} + C_nx^n + \ldots + C_1x + C_0 \ldots (4).$$

If now in (4) we put $\theta = 0$, we have

$$\left(\frac{d}{dx}\right)^{-n}\log x = (-1)^{n+1}\,\Gamma\overline{-n}.x^n + C_nx^n + \ldots\ldots + C_1x + C_0.$$

But we know *a priori* that this process introduces discontinuity into our results; which may be removed by supposing C_n divided, as before, into two parts C_n' and C_n, the former *discontinuous* and the latter *continuous*; whence, after an obvious reduction,

$$\left(\frac{d}{dx}\right)^{-n}\log x = \frac{x^n}{\Gamma\overline{n+1}}(\log x - A_n) + C_nx^n + \ldots\ldots + C_1x + C_0 \ldots (5).$$

By pursuing our investigations in this way, we obtain *complete* general differential expressions, including the complementary function. It is to be observed that, although in obtaining the *principal* function the order of operation is indifferent, yet it is not so when the complementary function is concerned. Thus, taking

$$\left(\frac{d}{dx}\right)^{-n}\log x = (-1)^{n+1}\,\Gamma\overline{-n}\,x^n + C_n'x^n + C_nx^n + \ldots\ldots C_1x + C_0,$$

and operating on it by $\left(\frac{d}{dx}\right)^{\theta}$, we obtain

$$\left(\frac{d}{dx}\right)^{-n+\theta}\log x = (-1)^{n+\theta+1}\,\Gamma\overline{-n+\theta}.x^{n-\theta},$$

the principal function only; while the complementary function, consisting of terms of the form C_rx^r, where r is a positive integer, wholly disappears by the operation. It is, however, useful to remark, that in this case the complementary function is indicated by the negative integer in the index of differentiation, being given by $\left(\frac{d}{dx}\right)^{-n}0$. The same thing is also indicated by $\Gamma\overline{-n+\theta}$ in the principal function, regard being had to the negative integer only under the symbol Γ while θ

the fractional part is positive. In the same way, we are able to find an indication of the complementary function peculiar to $\left(\frac{d}{dx}\right)^{-n-\theta}\log x$; for by a slight artifice we may resolve the quantity $(-n-\theta)$, under the functional symbol Γ in the principal function, into two parts, the one a *negative integer* and the other a *positive fraction;* thus $\Gamma\overline{-n-\theta} = \Gamma\overline{-n-1+(1-\theta)}$, giving $\left(\frac{d}{dx}\right)^{-n-1}.0$ for the complementary function, as in (4). Again, when in the principal function of $\left(\frac{d}{dx}\right)^{-n+\theta}\log x$ we put $n=0$, we have $\Gamma\theta$ in said function, without any indication of a complementary function, as in (2); also, when $n=0$ in the principal function of $\left(\frac{d}{dx}\right)^{-n-\theta}\log x$, we have $\Gamma\overline{-\theta} = \Gamma\overline{-1+(1-\theta)}$, giving by indication $\left(\frac{d}{dx}\right)^{-1}0 = C$; so that

$$\left(\frac{d}{dx}\right)^{-\theta}\log x = (-1)^{1-\theta}\,\Gamma\overline{-\theta}.x^{\theta} + C\,\ldots\ldots\ldots(6),$$

a result involved in equation (4).

It is important at our present stage to mark well this conventional method of inferring the form of the complementary from the principal function; as the same method will afterwards receive a remarkable extension with a very slight modification only.

The property in (2) could have been obtained in a very different way, to which it will be useful now to advert. For let us at once perform the arbitrary operation $\left(\frac{d}{dx}\right)^{\theta-1}$ on $\frac{1}{x}$, when

$$\left(\frac{d}{dx}\right)^{\theta-1}\frac{1}{x} = (-1)^{\theta+1}\frac{\Gamma\theta}{x^{\theta}} + C;$$

the order of the composite operation being supposed to be first by $\left(\frac{d}{dx}\right)^{\theta}$ and then by $\left(\frac{d}{dx}\right)^{-1}$, which last introduces the constant C. But again

$$\left(\frac{d}{dx}\right)^{\theta-1}\frac{1}{x} = \left(\frac{d}{dx}\right)^{\theta}\int\frac{dx}{x} = \left(\frac{d}{dx}\right)^{\theta}(\log x + C') = \left(\frac{d}{dx}\right)^{\theta}\log x;$$

hence $$\left(\frac{d}{dx}\right)^{\theta}\log x = (-1)^{\theta+1}\frac{\Gamma\theta}{x^{\theta}} + C.$$

But here we have no direct method of determining the value of the constant C; though by considering it in connexion with θ when just passing into unity, its limiting value, it is easy to infer that $C = 0$. Here, however, we have received one important element of instruction. It is now obvious that the performance of the arbitrary operation $\left(\frac{d}{dx}\right)^{\theta-1}$ on functions of the form $\frac{1}{x}$ and $\frac{1}{x+a}$ admits of no *apparent* arbitrary constant. But from what has been already shewn, it is evident that, in this *singular* case, a discontinuous constant has really appeared and disappeared in the composite operation.

This branch of our subject leads us naturally to advert to those instances of fractional differentiation, which analysis has already introduced into the department of definite integrals. Here a process of the following nature has been admitted as legitimate. Take, for example, the integral

$$\int_0^\infty e^{-ax^n} x^{n-1} dx = \frac{1}{na},$$

n being integer; and differentiate $(m-1)$ times successively according to a: when (rejecting the factor $(-1)^{m-1}$ common to both sides) there results

$$\int_0^\infty e^{-ax^n} x^{mn-1} dx = \frac{\frac{1}{n}\Gamma m}{a^m};$$

and this is also admitted to be true for *all* values of $m > 0$, without any constant of correction; so that we may put $m = \frac{1}{n}$, when

$$\int_0^\infty e^{-ax^n} dx = \frac{\frac{1}{n}\Gamma\frac{1}{n}}{a^{\frac{1}{n}}};$$

and when $a = 1$, $$\int_0^\infty e^{-x^n} dx = \frac{1}{n}\Gamma\frac{1}{n}.$$

It is now manifest that the *rationale* of this process depends upon our foregoing deduction, that the arbitrary operation $\left(\frac{d}{dx}\right)^{\theta-1}$, performed upon functions of the form $\frac{1}{x}$, admits of no constant of integration.

I now proceed to deduce from the foregoing some other complete differential expressions. For the sake of con-

venience, change θ into p in (6), when

$$\left(\frac{d}{dx}\right)^{-p} \log x = (-1)^{1-p} \Gamma\overline{-p}\, x^p + C.$$

First, let $\theta > p$, and operate on this with $\left(\frac{d}{dx}\right)^{\theta}$, when

$$\left(\frac{d}{dx}\right)^{\theta-p} \log x = (-1)^{1-p} \Gamma\overline{-p} \left(\frac{d}{dx}\right)^{\theta} x^p, \text{ since } \left(\frac{d}{dx}\right)^{\theta} C = 0\,;$$

but $$\left(\frac{d}{dx}\right)^{\theta-p} \log x = (-1)^{\theta-p+1} \frac{\Gamma\overline{\theta-p}}{x^{\theta-p}} \text{ by (2)}\,;$$

hence $$\left(\frac{d}{dx}\right)^{\theta} x^p = (-1)^{\theta} \frac{\Gamma\overline{-p+\theta}}{\Gamma\overline{-p}}\, x^{p-\theta} \,\ldots\ldots\ldots \,(7).$$

This form obviously admits of no arbitrary constant; and here it is to be observed that $\theta - p > 0$, while p and θ are any positive proper fractions.

But secondly, when $\theta < p$, we have

$$\left(\frac{d}{dx}\right)^{\theta-p} \log x = (-1)^{\theta-p+1} \Gamma\overline{\theta-p}.x^{p-\theta} + C \text{ by (6)};$$

so that in this case

$$\left(\frac{d}{dx}\right)^{\theta} x^p = (-1)^{\theta} \frac{\Gamma\overline{-p+\theta}}{\Gamma\overline{-p}}\, x^{p-\theta} + C \,\ldots\ldots \,(8);$$

where it is to be observed that $\theta - p < 0$.

Between these two forms lies the singular value

$$\left(\frac{d}{dx}\right)^{\theta} x^{\theta} = (-1)^{\theta} \Gamma 0 . x^0 + C,$$

being a case analogous to that of the integral $\int x^m dx$ when $m = -1$. Its real value might be deduced from (8) by the method of evanishing fractions; but we can obtain it directly by operating on (6) with $\left(\frac{d}{dx}\right)^{\theta}$, whence

$$\log x = (-1)^{1-\theta} \Gamma\overline{-\theta} \left(\frac{d}{dx}\right)^{\theta} x^{\theta}, \text{ since } \left(\frac{d}{dx}\right)^{\theta} C = 0\,;$$

and $$\left(\frac{d}{dx}\right)^{\theta} x^{\theta} = (-1)^{\theta+1} \frac{\log x}{\Gamma\overline{-\theta}} = (-1)^{\theta}. \frac{\theta.\log x}{\Gamma\overline{1-\theta}} \,\ldots\ldots \,(9).$$

After the same manner, let us change θ into p in (2); then

$$\left(\frac{d}{dx}\right)^{p} \log x = (-1)^{p+1} \lceil p \,.\, x^{-p}.$$

Suppose now $\theta > p$, and operate with $\left(\frac{d}{dx}\right)^{-\theta}$; then

$$\left(\frac{d}{dx}\right)^{p-\theta} \log x = (-1)^{p+1} \lceil p \left(\frac{d}{dx}\right)^{-\theta} x^{-p};$$

but $$\left(\frac{d}{dx}\right)^{p-\theta} \log x = (-1)^{p-\theta+1} \lceil\overline{p-\theta}.x^{-p+\theta} \text{ by } (2);$$

hence $$\left(\frac{d}{dx}\right)^{-\theta} x^{-p} = (-1)^{-\theta} \frac{\lceil\overline{p-\theta}}{\lceil p} x^{-p+\theta} \ldots\ldots\ldots (10),$$

without an arbitrary constant, and where $p - \theta > 0$. In the other case, in which $\theta > p$, we have

$$\left(\frac{d}{dx}\right)^{p-\theta} \log x = (-1)^{p-\theta+1} \lceil\overline{p-\theta}.x^{-p+\theta} + C \text{ by } (6);$$

whence $$\left(\frac{d}{dx}\right)^{-\theta} x^{-p} = (-1)^{-\theta} \frac{\lceil\overline{p-\theta}}{\lceil p} x^{-p+\theta} + C \ldots\ldots (11),$$

where $p - \theta < 0$. Intermediate to these two forms lies also a singular value when $\theta = p$, to which the same remarks apply, as before. But we can obtain the real value by operating on (2) with $\left(\frac{d}{dx}\right)^{-\theta}$; whence, after a slight reduction,

$$\left(\frac{d}{dx}\right)^{-\theta} x^{-\theta} = (-1)^{1-\theta} \frac{\log x}{\lceil\theta} + C \ldots\ldots\ldots (12).$$

The constant appears here in consequence of the complementary operation

$$\left(\frac{d}{dx}\right)^{-\theta}.0 = D^{-\theta}.0 = \frac{D^{1-\theta}.0}{D} = \frac{0}{D} = D^{-1}0 = C.$$

In the particular case when $\theta = 1$, we have, as we ought,

$$\int \frac{dx}{x} = \log x + C.$$

It may not be improper to give some verifications of (9) and (12). For this purpose, let us take $y = x^{\frac{1}{2}}$, and putting $D = \frac{d}{dx}$, we have $D^{\frac{1}{2}}y = (-1)^{\frac{1}{2}} \frac{\log x}{2\lceil\frac{1}{2}}$ by (9), so that

$$D^{\frac{1}{2}}y - y = (-1)^{\frac{1}{2}} \frac{\log x}{2\lceil\frac{1}{2}} - x^{\frac{1}{2}}.$$

Being thus assured of the synthesis of the equation thus formed, we may now proceed by analysis to its solution. Hence

$$y = \frac{1}{D^{\frac{1}{2}} - 1}\left\{(-1)^{\frac{1}{2}}\frac{\log x}{2\lceil\overline{\tfrac{1}{2}}} - x^{\frac{1}{2}} + 0\right\},$$

$$= \frac{D^{\frac{1}{2}} + 1}{D - 1}\left\{(-1)^{\frac{1}{2}}\frac{\log x}{2\lceil\overline{\tfrac{1}{2}}} - x^{\frac{1}{2}} + 0\right\},$$

$$= \frac{1}{D - 1}\left\{\frac{1}{2x^{\frac{1}{2}}} - x^{\frac{1}{2}} + 0\right\} \text{ by (2) and (9),}$$

$$= e^{x}\int\left(\frac{1}{2x^{\frac{1}{2}}} - x^{\frac{1}{2}}\right)e^{-x}dx + Ce^{x};$$

or
$$y = e^{x}(e^{-x}x^{\frac{1}{2}}) + Ce^{x} = x^{\frac{1}{2}} + Ce^{x},$$

which is the complete solution. As a simple verification of (9) and (12) together, take $y = x^{\frac{1}{2}}$ as before, when

$$2.D^{\frac{1}{2}}y - D^{-\frac{1}{2}}y^{-1} = c, \text{ by (9) and (12).}$$

To solve this equation, operate with $D^{\frac{1}{2}}$, when

$$2Dy - y^{-1} = 0, \text{ or } 2\frac{dy}{dx} - \frac{1}{y} = 0;$$

giving $y = (x + C)^{\frac{1}{2}}$ for the complete solution. When $C = 0$, then $y = x^{\frac{1}{2}}$, agreeing with the primary assumption.

If we now examine the principal function in (7), (8), (10), (11), restricting our attention to the numerator only under the symbol $\lceil$, it is easily seen that the conventional rule before given, for inferring the complementary from the principal function, holds exactly. Thus in (7) from $\lceil\overline{-p+\theta}$ we have $(\theta - p) > 0$ with no constant; in (8) we have $(\theta - p) < 0$, so that $\lceil\overline{-p+\theta} = \lceil\overline{-1 + (1 - p + \theta)}$, indicating one constant; and so of (10) and (11). A singular case presents itself in (12); but there the constant is indicated by the differential index.

It is not the object of this paper to enter at large upon the subject of the complementary function in general differentiation; a subject so essential to the proper interpretation of its processes, as to demand a separate and methodical investigation.

I now proceed to find the general differential coefficients of $\sin ax$ and $\cos ax$. Taking $\cos ax$ first, we have

$$2\cos ax = e^{ax\sqrt{(-1)}} + e^{-ax\sqrt{(-1)}}.$$

Operate with $\left(\dfrac{d}{dx}\right)^\theta$ on both sides, when

$$\begin{aligned}2\left(\frac{d}{dx}\right)^\theta \cos ax &= (-1)^{\frac{1}{2}\theta} a^\theta e^{ax\sqrt{(-1)}} + (-1)^\theta (-1)^{\frac{1}{2}\theta} a^\theta e^{-ax\sqrt{(-1)}},\\ &= (-1)^\theta a^\theta e^{ax\sqrt{(-1)}} (-1)^{-\frac{1}{2}\theta} + (-1)^\theta a^\theta e^{-ax\sqrt{(-1)}} (-1)^{\frac{1}{2}\theta},\\ &= (-1)^\theta a^\theta \left\{e^{(ax-\theta\frac{1}{2}\pi)\sqrt{(-1)}} + e^{-(ax-\theta\frac{1}{2}\pi)\sqrt{(-1)}}\right\},\\ &= 2(-1)^\theta a^\theta \cos(ax - \theta\tfrac{1}{2}\pi);\end{aligned}$$

hence $$\left(\frac{d}{dx}\right)^\theta \cos ax = (-1)^\theta a^\theta \cos(ax - \theta\tfrac{1}{2}\pi). \;\ldots\ldots(13).$$

Again, $\quad 2\sqrt{(-1)}\sin ax = e^{ax\sqrt{(-1)}} - e^{-ax\sqrt{(-1)}};$

and following the same process, we have finally

$$\left(\frac{d}{dx}\right)^\theta \sin ax = (-1)^\theta a^\theta \sin(ax - \theta\tfrac{1}{2}\pi) \;\ldots\ldots(14).$$

These are the expressions required.

I now go on to give a few verifications of the preceding general differential forms.

1. Since $$\int_0 \frac{dx}{a+x} = \log(a+x) - \log a;$$

let us operate with $\left(\dfrac{d}{da}\right)^\theta$, when

$$(-1)^\theta \Gamma\overline{\theta+1} \int_0 \frac{dx}{(a+x)^{\theta+1}} = (-1)^{\theta+1}\Gamma\theta\left\{\frac{1}{(a+x)^\theta} - \frac{1}{a^\theta}\right\},$$

and $$\int_0 \frac{dx}{(a+x)^{\theta+1}} = \frac{1}{\theta}\left\{\frac{1}{a^\theta} - \frac{1}{(a+x)^\theta}\right\}.$$

This result is manifestly true for all finite values of θ.

2. If in the known integral $\int_0^\infty \dfrac{dx}{1+x^2} = \frac{1}{2}\pi$, we write $a^{\frac{1}{2}}x$ for x, the limits will remain unchanged; so that

$$a^{\frac{1}{2}}\int_0^\infty \frac{dx}{1+ax^2} = \tfrac{1}{2}\pi, \quad \text{or} \int_0^\infty \frac{x^{-2}\,dx}{x^{-2}+a} = \frac{\pi}{2a^{\frac{1}{2}}}.$$

Operate with $\left(\dfrac{d}{da}\right)^\theta$ and then divide by $(-1)^\theta\,\Gamma\overline{\theta+1}$, when

$$\int_0^\infty \frac{x^{-2}dx}{(x^{-2}+a)^{\theta+1}} = \frac{\Gamma\overline{\theta+\frac{1}{2}}}{\Gamma\overline{\theta+1}} \cdot \frac{\pi^{\frac{1}{2}}}{2a^{\theta+\frac{1}{2}}},$$

or $$\int_0^\infty \frac{x^{2\theta}dx}{(1+ax^2)^{\theta+1}} = \frac{\Gamma\overline{\theta+\frac{1}{2}}}{\Gamma\overline{\theta+1}} \cdot \frac{\pi^{\frac{1}{2}}}{2a^{\theta+\frac{1}{2}}}.$$

Let $a = 1$ and $\theta = \frac{1}{2}$, then $\int_0^\infty \frac{x\,dx}{(1+x^2)^{\frac{3}{2}}} = 1$; as we know also from common integration.

3. Since $a.\int_0 \frac{dx}{b+ax} = \log(b+ax) - \log b$,

or $$a.\int \frac{x^{-1}dx}{bx^{-1}+a} = \log(bx^{-1}+a) - \log bx^{-1};$$

operate with $\left(\frac{d}{da}\right)^\theta$ on both sides (employing Leibnitz's theorem on the first side, when the operation will terminate after the first two terms); when, after division by $(-1)^\theta$, we have

$$\lceil \theta+1 . a.\int_0 \frac{x^{-1}dx}{(bx^{-1}+a)^{\theta+1}} - \theta \lceil \theta \int_0 \frac{x^{-1}dx}{(bx^{-1}+a)^\theta} = -\frac{\lceil \theta}{(bx^{-1}+a)^\theta},$$

and $$a.\int_0 \frac{x^\theta dx}{(b+ax)^{\theta+1}} = \int_0 \frac{x^{\theta-1}dx}{(b+ax)^\theta} - \frac{x^\theta}{\theta.(b+ax)^\theta}.$$

This result is easily seen to be true for all finite values of θ the index of differentiation, being also derivable from the integration by parts of the function $\frac{x^\theta}{(b+ax)^\theta}$.

4. Since $\int_0^\infty \sin ax\,dx = \frac{1}{a}$, operate by $\left(\frac{d}{da}\right)^{\theta-1}$ and divide by $(-1)^{\theta-1}$, when

$$\int_0^\infty dx\,x^{\theta-1}\sin(ax - \theta\tfrac{1}{2}\pi + \tfrac{1}{2}\pi) = \frac{\lceil \theta}{a^\theta},$$

or $$\int_0^\infty dx\,x^{\theta-1}\cos(ax - \theta\tfrac{1}{2}\pi) = \frac{\lceil \theta}{a^\theta},$$

or $$\cos(\theta\tfrac{1}{2}\pi)\int_0^\infty dx\,x^{\theta-1}\cos ax + \sin(\theta\tfrac{1}{2}\pi)\int_0^\infty dx\,x^{\theta-1}\sin ax = \frac{\lceil \theta}{a^\theta} \dots (15).$$

5. Since $\int_0^\infty \cos ax\,dx = 0$; operate with $\left(\frac{d}{da}\right)^{\theta-1}$, and

$$\int_0^\infty dx\,x^{\theta-1}\cos(ax - \theta\tfrac{1}{2}\pi + \tfrac{1}{2}\pi) = 0,$$

or $$\int_0^\infty dx\,x^{\theta-1}\sin(ax - \theta\tfrac{1}{2}\pi) = 0;$$

whence

$$\cos(\theta\tfrac{1}{2}\pi)\int_0^\infty dx\,x^{\theta-1}\sin ax = \sin(\theta\tfrac{1}{2}\pi)\int_0^\infty dx\,x^{\theta-1}\cos ax \ldots(16).$$

By comparing (15) and (16) we easily deduce the well-known integrals

$$\int_0^\infty dx\,x^{\theta-1}\cos ax = \frac{\Gamma\theta}{a^\theta}\cos(\theta\tfrac{1}{2}\pi),$$

and
$$\int_0^\infty dx\,x^{\theta-1}\sin ax = \frac{\Gamma\theta}{a^\theta}\sin(\theta\tfrac{1}{2}\pi).$$

6. Taking the integral

$$\int_0^\infty \frac{\cos ax\,dx}{1+x^2} = \tfrac{1}{2}\pi e^{-a},$$

operate with $\left(\frac{d}{da}\right)^\theta$ and divide by $(-1)^\theta$, when

$$\int_0^\infty \frac{\cos(ax-\theta\tfrac{1}{2}\pi)\,x^\theta dx}{1+x^2} = \tfrac{1}{2}\pi e^{-a};$$

put $a=0$, $\cos(-\theta\tfrac{1}{2}\pi)\int_0^\infty \frac{x^\theta dx}{1+x^2} = \tfrac{1}{2}\pi$; or $\int_0^\infty \frac{x^\theta dx}{1+x^2} = \frac{\pi}{2\cos(\theta\tfrac{1}{2}\pi)}$.

And since θ may be any proper fraction, put $1-\theta$ for θ in last form, when

$$\int_0^\infty \frac{x^{1-\theta}dx}{1+x^2} = \frac{\pi}{2\sin(\theta\tfrac{1}{2}\pi)}.$$

We know these results to be true.

7. By common integration

$$\int_0^\infty e^{-ax^n\sqrt{(-1)}}x^{n-1}dx = \frac{(-1)^{-\frac{1}{2}}}{na}.$$

Perform the operation $\left(\frac{d}{da}\right)^{\theta-1}$ on both sides and divide by $(-1)^{\frac{3}{2}\theta-1}$, when

$$\int_0^\infty e^{-ax^n\sqrt{(-1)}}x^{n\theta-1}dx = (-1)^{-\frac{1}{2}\theta}.\frac{\Gamma\theta}{na^\theta};$$

put $\theta = \frac{1}{n}$, $\quad \int_0^\infty e^{-ax^n\sqrt{(-1)}}dx = (-1)^{-\frac{1}{2n}}.\frac{\frac{1}{n}\Gamma\frac{1}{n}}{a^{\frac{1}{n}}}.$

But $\int_0^\infty e^{-ax^n\sqrt{(-1)}}dx = \int_0^\infty \cos(ax^n)\,dx - \sqrt{(-1)}\int_0^\infty \sin(ax^n)\,dx,$

and $$\frac{\frac{1}{n}\Gamma\frac{1}{n}}{a^{\frac{1}{n}}}(-1)^{-\frac{1}{2n}} = \frac{\frac{1}{n}\Gamma\frac{1}{n}}{a^{\frac{1}{n}}}.\left(\cos\frac{\pi}{2n} - \sqrt{(-1)}\sin\frac{\pi}{2n}\right);$$

hence, equating the real and imaginary terms of the second members of these equalities, we have the following known integrals:

$$\int_0^\infty \cos(ax^n)\,dx = \frac{\frac{1}{n}\Gamma\frac{1}{n}}{a^{\frac{1}{n}}}.\cos\left(\frac{\pi}{2n}\right),$$

and $$\int_0^\infty \sin(ax^n)\,dx = \frac{\frac{1}{n}\Gamma\frac{1}{n}}{a^{\frac{1}{n}}}.\sin\left(\frac{\pi}{2n}\right).$$

Longside, Mintlaw, Oct. 24, 1848.

ON THE ATTRACTION OF ELLIPSOIDS (JACOBI'S METHOD).

By Arthur Cayley.

In a letter published in 1846 in *Liouville's Journal* (tom. XI. p. 341) Jacobi says, "Il y a quatorze ans, je me suis posé le problème de chercher l'attraction d'un ellipsoïde homogène exercée sur un point extérieur quelconque par une méthode analogue à celle employée par Maclaurin par rapport aux points situés dans les axes principaux. J'y suis parvenu par trois substitutions consécutives. La première est une transformation de coordonnées; par la seconde le radical $\sqrt{(1 - m^2\sin^2\beta\cos^2\psi - n^2\sin^2\beta\sin^2\psi)}$ qui entre dans la double integrale transformée est rendu rationnel au moyen de la double substitution

$$m\sin\beta\cos\psi = \sin\eta\cos\theta,\quad m\sin\beta\sin\psi = \sin\eta\sin\theta;$$

la troisieme est encore une transformation de coordonnées. La recherche du sens géométrique de ces trois substitutions m'a conduit à approfondir la théorie des surfaces confocales par rapport auxquelles je découvris quantitè de beaux théorèmes dont je communiquai quelques-uns des principaux à M. Steiner. Considérons l'ellipsoïde confocal mené

par le point attiré P et le point p de l'ellipsoïde proposé, conjugué à P. Soient Q et q deux autres points conjugués quelconques situées respectivement sur l'ellipsoïde extérieur et intérieur. Menons de P un premier cône tangent à l'ellipsoïde intérieur, de p un second cône tangent à l'ellipsoïde extérieur. Ce dernier, tout imaginaire qu'il est, a ses trois axes réels (ainsi que ses deux droites focales). La première substitution ramène les axes de l'ellipsoïde à ceux du premier cône (c'est la substitution employée par Poisson, mais que j'avais antérieurement traitée et même etendue à un nombre quelconque de variables dans le memoire *De binis Functionibus homogeneis, &c.*). Par la seconde substitution les angles que la droite Pq forme avec les axes du premier cône sont ramenés aux angles que la droite pQ forme avec les axes du second. Par la dernière substitution, on retourne de ces axes aux axes de l'ellipsoïde. La seconde substitution répond à un théorème de géométrie remarquable savoir que 'Les cosinus des angles que la droite Pq forme avec deux des axes du premier cône sont en raison constante avec les cosinus des angles que la droite pQ forme avec deux des axes du second cône; ces deux axes sont les tangents situés respectivement dans les sections de plus grande et de moindre courbure de chaque ellipsoïde, le troisième axe étant la normale à l'ellipsoïde.' Tout cela semble difficile à etablir par la synthèse."

The object of this paper is to develope the above method of finding the attraction of an ellipsoid.

Consider an exterior ellipsoid, the squared semiaxes of which are $f+u$, $g+u$, $h+u$. And an interior ellipsoid, the squared semiaxes of which are $f+\bar{u}$, $g+\bar{u}$, $h+\bar{u}$. Let u, p, q be the elliptic coordinates of a point P on the exterior ellipsoid, the elliptic coordinates of the corresponding point $\bar{P}$ on the interior ellipsoid will be $\bar{u}, p, q$, and if a, b, c and $\bar{a}, \bar{b}, \bar{c}$ represent the ordinary coordinates of these points (the principal axes being the axes of coordinates), we have

$$a^2=\frac{(f+u)(f+q)(f+r)}{(f-g)(f-h)}, \qquad \bar{a}^2=\frac{(f+\bar{u})(f+q)(f+r)}{(f-g)(f-h)},$$

$$b^2=\frac{(g+u)(g+q)(g+r)}{(g-h)(g-f)}, \qquad \bar{b}^2=\frac{(g+\bar{u})(g+q)(g+r)}{(g-h)(g-f)},$$

$$c^2=\frac{(h+u)(h+q)(h+r)}{(h-g)(h-f)}, \qquad \bar{c}^2=\frac{(h+\bar{u})(h+q)(h+r)}{(h-f)(h-g)}.$$

I form the systems of equations

$$a_1^2 = \frac{(u+f)(u+g)(u+h)}{(u-q)(u-r)}, \qquad \bar{a}_1^2 = \frac{(\bar{u}+f)(\bar{u}+g)(\bar{u}+h)}{(\bar{u}-q)(\bar{u}-r)},$$

$$b_1^2 = \frac{(q+f)(q+g)(q+h)}{(q-r)(q-u)}, \qquad \bar{b}_1^2 = \frac{(q+f)(q+g)(q+h)}{(\bar{q}-r)(q-\bar{u})},$$

$$c_1^2 = \frac{(r+f)(r+g)(r+h)}{(r-u)(r-q)}, \qquad \bar{c}_1^2 = \frac{(r+f)(r+g)(r+h)}{(r-\bar{u})(r-q)}.$$

$$\alpha = \frac{a_1 a}{f+u}, \quad \beta = \frac{a_1 b}{g+u}, \quad \gamma = \frac{a_1 c}{h+u},$$

$$\alpha' = \frac{b_1 a}{f+q}, \quad \beta' = \frac{b_1 b}{g+q}, \quad \gamma' = \frac{b_1 c}{h+q},$$

$$\alpha'' = \frac{c_1 a}{f+r}, \quad \beta'' = \frac{c_1 b}{g+r}, \quad \gamma'' = \frac{c_1 c}{h+r}.$$

$$\bar{\alpha} = \frac{\bar{a}_1 \bar{a}}{f+\bar{u}}, \quad \bar{\beta} = \frac{\bar{a}_1 \bar{b}}{g+\bar{u}}, \quad \bar{\gamma} = \frac{\bar{a}_1 \bar{c}}{h+\bar{u}},$$

$$\bar{\alpha}' = \frac{\bar{b}_1 \bar{a}}{f+q}, \quad \bar{\beta}' = \frac{\bar{b}_1 \bar{b}}{g+q}, \quad \bar{\gamma}' = \frac{\bar{b}_1 \bar{c}}{h+q},$$

$$\bar{\alpha}'' = \frac{\bar{c}_1 \bar{a}}{f+r}, \quad \bar{\beta}'' = \frac{\bar{c}_1 \bar{b}}{g+r}, \quad \bar{\gamma}'' = \frac{\bar{c}_1 \bar{c}}{h+r}.$$

And then writing

$$X = \alpha X_1 + \alpha' Y_1 + \alpha'' Z_1, \qquad \bar{X} = \bar{\alpha}\bar{X}_1 + \bar{\alpha}'\bar{Y}_1 + \bar{\alpha}''\bar{Z}_1,$$

$$Y = \beta X_1 + \beta' Y_1 + \alpha'' Z_1, \qquad \bar{Y} = \bar{\beta}\bar{X}_1 + \bar{\beta}'\bar{Y}_1 + \bar{\beta}''\bar{Z}_1,$$

$$Z = \gamma X_1 + \gamma' Y_1 + \gamma'' Z_1, \qquad \bar{Z} = \bar{\gamma}\bar{X}_1 + \bar{\gamma}'\bar{Y}_1 + \bar{\gamma}''\bar{Z}_1.$$

If X, Y, Z are the cosines of the inclinations of a line $P\bar{Q}$ to the principal axes of the ellipsoids X_1, Y_1, Z_1 will be the cosines of the inclinations of this line to the principal axes of the cone having P for its vertex, and circumscribed about the interior ellipsoid. In like manner, $\bar{X}$, $\bar{Y}$, $\bar{Z}$ being the cosines of the inclinations of a line $\bar{P}Q$ to the principal axes of the ellipsoids $\bar{X}_1$, $\bar{Y}_1$, $\bar{Z}_1$ will be the cosines of the inclinations of this line to the principal axes of the cone having $\bar{P}$ for its vertex and circumscribed about the exterior ellipsoid. Assuming that the points Q, $\bar{Q}$ are situated upon the exterior and interior ellipsoids respectively, suppose that X_1, Y_1, Z_1 and $\bar{X}_1$, $\bar{Y}_1$, $\bar{Z}_1$ are connected by the equivalent systems of equations,

$$X_1 = \sqrt{(u-\bar{u})}\sqrt{\left(\frac{\bar{X}_1^2}{u-\bar{u}} + \frac{\bar{Y}_1^2}{u-q} + \frac{\bar{Z}_1^2}{u-r}\right)},$$

$$Y_1 = \sqrt{\left(\frac{\bar{u}-q}{u-q}\right)}\,\bar{Y}_1,$$

$$Z_1 = \sqrt{\left(\frac{\bar{u}-r}{u-r}\right)}\,\bar{Z}_1.$$

$$\bar{X}_1 = \sqrt{(\bar{u}-u)}\sqrt{\left(\frac{X_1^2}{\bar{u}-u} + \frac{Y_1^2}{\bar{u}-q} + \frac{Z_1^2}{\bar{u}-r}\right)},$$

$$\bar{Y}_1 = \sqrt{\left(\frac{u-q}{\bar{u}-q}\right)}\,Y_1,$$

$$\bar{Z}_1 = \sqrt{\left(\frac{u-r}{\bar{u}-r}\right)}\,Z_1.$$

Then it will presently be shewn that the points Q, $\bar{Q}$ are corresponding points, which will prove the geometrical theorem of Jacobi. Before proceeding further it will be convenient to notice the formulæ

$$1 - \frac{a^2}{f+\bar{u}} - \frac{b^2}{g+\bar{u}} - \frac{c^2}{h+\bar{u}} = \frac{\bar{u}-u}{\bar{a}_1^2},$$

$$\frac{Xa}{f+\bar{u}} + \frac{Yb}{g+\bar{u}} + \frac{Zc}{h+\bar{u}} = \frac{\bar{u}-u}{\bar{a}_1^2}\left(\frac{X_1a_1}{\bar{u}-u} + \frac{Y_1b_1}{\bar{u}-q} + \frac{Z_1c_1}{\bar{u}-r}\right),$$

$$\left(\frac{Xa}{f+\bar{u}} + \frac{Yb}{g+\bar{u}} + \frac{Zc}{h+\bar{u}}\right)^2 + \frac{\bar{u}-u}{\bar{a}_1^2}\left(\frac{X^2}{f+\bar{u}} + \frac{Y^2}{g+\bar{u}} + \frac{Z^2}{h+\bar{u}}\right)$$
$$= \frac{\bar{u}-u}{\bar{a}_1^2}\left(\frac{X_1^2}{\bar{u}-u} + \frac{Y_1^2}{\bar{u}-q} + \frac{Z_1^2}{\bar{u}-r}\right) = \frac{\bar{X}_1^2}{\bar{a}_1^2}.$$

And the corresponding ones

$$1 - \frac{\bar{a}^2}{f+u} - \frac{\bar{b}^2}{g+u} - \frac{\bar{c}^2}{h+u} = \frac{u-\bar{u}}{a_1^2}.$$

$$\frac{\bar{X}\bar{a}}{f+u} + \frac{\bar{Y}\bar{b}}{g+u} + \frac{\bar{Z}\bar{c}}{h+u} = \frac{u-\bar{u}}{a_1^2}\left(\frac{\bar{X}_1\bar{a}_1}{u-\bar{u}} + \frac{\bar{Y}_1\bar{b}_1}{u-q} + \frac{\bar{Z}_1\bar{c}_1}{u-r}\right),$$

$$\left(\frac{\bar{X}\bar{a}}{f+u} + \frac{\bar{Y}\bar{b}}{g+u} + \frac{\bar{Z}\bar{c}}{h+u}\right)^2 + \frac{u-\bar{u}}{a_1^2}\left(\frac{\bar{X}^2}{f+u} + \frac{\bar{Y}^2}{g+u} + \frac{\bar{Z}^2}{h+u}\right)$$
$$= \frac{u-\bar{u}}{a_1^2}\left(\frac{\bar{X}_1^2}{u-\bar{u}} + \frac{\bar{Y}_1^2}{u-q} + \frac{\bar{Z}_1^2}{u-r}\right) = \frac{X_1^2}{a_1^2}.$$

The coordinates of the point $\overline{Q}$ are obviously $a+\rho X$, $b+\rho Y$, $c+\rho Z$ (where $\rho = P\overline{Q}$). Substituting these values in the equation of the interior ellipsoid, we obtain

$$\rho^2\left(\frac{X^2}{f+\bar{\bar{u}}}+\frac{Y^2}{g+\bar{\bar{u}}}+\frac{Z^2}{h+\bar{\bar{u}}}\right)+2\rho\left(\frac{Xa}{f+\bar{\bar{u}}}+\frac{Yb}{g+\bar{\bar{u}}}+\frac{Zc}{h+\bar{\bar{u}}}\right)$$
$$+\left(\frac{a^2}{f+\bar{\bar{u}}}+\frac{b^2}{g+\bar{\bar{u}}}+\frac{z^2}{c+\bar{\bar{u}}}-1\right)=0.$$

Reducing the coefficients of this equation by the formulæ first given, and omitting a factor $\dfrac{\bar{u}-u}{\bar{a}_1^2}$, we obtain

$$\left\{\frac{\bar{a}_1^2\overline{X}^2}{(\bar{u}-u)^2}-\left(\frac{X_1a_1}{\bar{\bar{u}}-u}+\frac{Y_1b_1}{\bar{u}_1-q}+\frac{Z_1c_1}{\bar{\bar{u}}-r}\right)^2\right\}\rho^2$$
$$+2\rho\left(\frac{X_1a_1}{\bar{\bar{u}}-u}+\frac{Y_1b_1}{\bar{\bar{u}}-q}+\frac{Z_1c_1}{\bar{\bar{u}}-r}\right)-1=0.$$

i.e. $$\frac{\bar{a}_1^2\overline{X}_1^2}{(\bar{\bar{u}}-u)^2}\rho^2=\left\{\rho\left(\frac{X_1a_1}{\bar{\bar{u}}-u}+\frac{Y_1b_1}{\bar{\bar{u}}-q}+\frac{Z_1c_1}{\bar{\bar{u}}-r}\right)-1\right\}^2,$$

or $$\rho=\frac{1}{\dfrac{X_1a_1-\overline{X}_1\bar{a}_1}{\bar{u}-u}+\dfrac{Y_1a_1}{\bar{\bar{u}}-q}+\dfrac{Z_1c_1}{\bar{u}-r}};$$

which is easily transformed into

$$\rho=\frac{\bar{u}-u}{a_1X_1-\bar{a}_1\overline{X}_1+\dfrac{(f+u)b_1Y_1-(f+\bar{u})\bar{b}_1\overline{Y}_1}{f+q}+\dfrac{(f+u)c_1Z_1-(f+\bar{u})\bar{c}_1\overline{Z}_1}{f+r}}.$$

And this form remaining unaltered when u and $\bar{u}$ are interchanged, it follows that if $\overline{P}Q=\bar{\rho}$, then $\rho=\bar{\rho}$, which is a known theorem. The value of ρ or $\bar{\rho}$ may however be expressed in a yet simpler form ; for, considering the expression

$$\frac{X}{\sqrt{(f+\bar{u})}}-\frac{\overline{X}}{\sqrt{(f+u)}}=\frac{a}{\sqrt{(f+\bar{u})}}\left\{\frac{a_1X_1}{f+u}+\frac{b_1Y_1}{f+q}+\frac{c_1Z_1}{f+r}\right\}$$
$$-\frac{\bar{a}}{\sqrt{(f+u)}}\left\{\frac{\bar{a}_1\overline{X}_1}{f+\bar{u}}+\frac{\bar{b}_1\overline{Y}_1}{f+q}+\frac{\bar{c}_1\overline{Z}_1}{f+r}\right\}$$

$$=\frac{-1}{\bar{u}-u}\left\{\frac{a}{\sqrt{(f+\bar{u})}}-\frac{\bar{a}}{\sqrt{(f+u)}}\right\}\times$$
$$\left\{a_1X_1-\bar{a}_1\overline{X}_1+\frac{(f+u)b_1Y_1-(f+\bar{u})\bar{b}_1\overline{Y}_1}{f+q}+\frac{(f+u)c_1Z_1-(f+\bar{u})\bar{c}_1\overline{Z}_1}{f+r}\right\},$$

we see that

$$\frac{X}{\sqrt{(f+\bar{u})}} - \frac{\bar{X}}{\sqrt{(f+u)}} = -\frac{1}{\rho}\left(\frac{a}{\sqrt{(f+\bar{u})}} - \frac{\bar{a}}{\sqrt{(f+u)}}\right),$$

and similarly

$$\frac{Y}{\sqrt{(f+\bar{u})}} - \frac{\bar{Y}}{\sqrt{(g+u)}} = -\frac{1}{\rho}\left(\frac{b}{\sqrt{(g+\bar{u})}} - \frac{\bar{b}}{\sqrt{(g+u)}}\right),$$

$$\frac{Z}{\sqrt{(h+\bar{u})}} - \frac{\bar{Z}}{\sqrt{(h+u)}} = -\frac{1}{\rho}\left(\frac{c}{\sqrt{(h+\bar{u})}} - \frac{\bar{c}}{\sqrt{(h+u)}}\right);$$

which are in fact the equations which express that Q and $\bar{Q}$ are corresponding points.

It is proper to remark that supposing, as we are at liberty to do, that P, $\bar{P}$ are situate in corresponding octants of the two ellipsoids, then if the curve of contact of the circumscribed cone having P for its vertex divide the surface of the interior ellipsoid into two parts $\bar{M}$, $\bar{N}$, of which the former lies contiguous to $\bar{P}$: also if the curve of intersection of the tangent plane at $\bar{P}$ divide the surface of the exterior ellipsoid into two parts M, N, of which M lies contiguous to the point P, then the different points of M, $\bar{M}$ correspond to each other, as do also the different points of N, $\bar{N}$.

We may now pass to the integral calculus problem. The attraction parallel to the axis of x is

$$A = \int \frac{x\,dx\,dy\,dz}{(x^2+y^2+z^2)^{\frac{3}{2}}},$$

the limits of the integration being given by

$$\frac{(x+a)^2}{f+\bar{u}} + \frac{(y+b)^2}{g+\bar{u}} + \frac{(z+c)^2}{h+\bar{u}} = 1.$$

Or putting $\qquad x = rX,\ y = rY,\ z = rZ,$

where X, Y, Z have the same signification, as before, we have

$$dx\,dy\,dz = r^2 dr\,dS,$$

and $\qquad A = \int X dr\,dS = \int \rho X dS,$

where ρ has the same signification as before: it will be convenient to leave the formula in this form, rather than to take at once the difference of the two values of ρ, but of course the integration is as in the ordinary methods to be performed so as to extend to the whole volume of the ellipsoid. The

expression dS denotes the differential of a spherical surface radius unity, and if θ, ϕ are the parameters by which the position of ρ is determined, we have

$$dS = \begin{vmatrix} X, & Y, & Z \\ \frac{dX}{d\theta}, & \frac{dY}{d\theta}, & \frac{dZ}{d\theta} \\ \frac{dX}{d\phi}, & \frac{dY}{d\phi}, & \frac{dZ}{d\phi} \end{vmatrix} d\theta\, d\phi.$$

In the present case

$$dS = dS_1 = \begin{vmatrix} X_1, & Y_1, & Z_1 \\ \frac{dX_1}{d\overline{Y}_1}, & \frac{dY_1}{d\overline{Y}_1}, & \frac{dZ_1}{d\overline{Y}_1} \\ \frac{dX_1}{d\overline{Z}_1}, & \frac{dY_1}{d\overline{Z}_1}, & \frac{dZ_1}{d\overline{Z}_1} \end{vmatrix} d\overline{Y}_1 d\overline{Z}_1.$$

Or from the values of X_1, Y_1, Z_1 in terms of $\overline{X}_1$, $\overline{Y}_1$, $\overline{Z}_1$ [observing that $\overline{X}_1$ must be replaced by its value $(\sqrt{1-\overline{Y}_1^2-\overline{Z}_1^2})$] we deduce

$$dS = \sqrt{\left\{\frac{(\bar{u}-q)(\bar{u}-r)}{(u-q)(u-r)}\right\}}\, \frac{1}{X_1}\, d\overline{Y}_1 d\overline{Z}_1.$$

But $d\overline{S} = d\overline{S}_1 = \dfrac{1}{\overline{X}_1}\, d\overline{Y}_1 d\overline{Z}_1$, whence

$$dS = \sqrt{\left\{\frac{(\bar{u}-q)(\bar{u}-r)}{(u-q)(u-r)}\right\}}\, \frac{\overline{X}_1 d\overline{S}}{X_1},$$

which shews that the corresponding elements of the spheres whose centres are P, $\overline{P}$, projected upon the tangent planes at P and $\overline{P}$ respectively, are in a constant ratio. It may be noticed also that if μ, $\bar{\mu}$ are the masses of the ellipsoids, the ratio in question

$$= \sqrt{\left\{\frac{(\bar{u}-q)(\bar{u}-r)}{(u-q)(u-r)}\right\}} = \frac{\bar{\mu} a_1}{\mu \bar{a}_1}.$$

We have thus

$$A = \sqrt{\left\{\frac{(\bar{u}-q)(\bar{u}-r)}{(u-q)(u-r)}\right\}} \int \frac{\bar{\rho}\, \overline{X}_1 X d\overline{S}}{X_1},$$

that is

$$A = \sqrt{\left\{\frac{(\bar{u}-q)(\bar{u}-r)}{(u-q)(u-r)}\right\}} \int \frac{\bar{\rho}\,(aX_1 + a'Y_1 + a''Z_1)\, \overline{X}_1 d\overline{S}}{X_1}.$$

The value which it will be convenient to use for $\bar{\rho}$ is that derived from the equation

$$\bar{\rho}^2\left(\frac{\bar{X}^2}{f+u}+\frac{\bar{Y}^2}{g+u}+\frac{\bar{Z}^2}{h+u}\right)+2\bar{\rho}\left(\frac{\bar{X}\bar{a}}{f+u}+\frac{\bar{Y}\bar{b}}{g+u}+\frac{\bar{Z}'\bar{c}}{h+u}\right)+\frac{\bar{u}-u}{a_1^{\,2}}=0,$$

with only the transformation of expressing the radical in terms of X_1, viz.

$$\rho=\frac{\dfrac{\bar{X}\bar{a}}{f+u}+\dfrac{\bar{Y}\bar{b}}{g+u}+\dfrac{\bar{Z}\bar{c}}{h+u}+\dfrac{1}{a_1}X_1}{\dfrac{\bar{X}^2}{f+u}+\dfrac{\bar{Y}^2}{g+u}+\dfrac{\bar{Z}^2}{h+u}};$$

substituting these values and observing that Y_1 and Z_1 are rational functions of $\bar{X}$, $\bar{Y}$, and $\bar{Z}$, but that $\bar{X}_1$ is a radical, and that in order to extend the integration to the whole ellipsoid, the values corresponding to the opposite signs of X_1 will require to be added, the quantity to be integrated is (omitting for the moment the exterior constant factor)

$$\frac{\left\{a\left(\dfrac{\bar{X}\bar{a}}{f+u}+\dfrac{\bar{Y}\bar{b}}{g+u}+\dfrac{\bar{Z}\bar{c}}{h+u}\right)+\dfrac{1}{a_1}(a'Y_1+a''Z_1)\right\}\bar{X}_1d\bar{S}}{\dfrac{\bar{X}^2}{f+u}+\dfrac{\bar{Y}^2}{g+u}+\dfrac{\bar{Z}^2}{h+u}},$$

the integration to be extended over the spherical area $\bar{S}$. Consider the quantity within { }, this is

$$a\left(\frac{\bar{X}\bar{a}}{f+u}+\frac{\bar{Y}\bar{b}}{g+u}+\frac{\bar{Z}\bar{c}}{h+u}\right)+\frac{a'}{a_1}\sqrt{\left(\frac{q-\bar{u}}{q-u}\right)}(\bar{\alpha}'\bar{X}+\bar{\beta}'\bar{Y}+\bar{\gamma}'\bar{Z})$$
$$+\frac{a''}{a_1}\sqrt{\left(\frac{r-\bar{u}}{r-u}\right)}(\bar{\alpha}''\bar{X}+\bar{\beta}''\bar{Y}+\bar{\gamma}''\bar{Z}).$$

The coefficients of $\bar{Y}$ and $\bar{Z}$ vanish, in fact that of $\bar{Y}$ is

$$\frac{a_1a}{f+u}\frac{\bar{b}}{g+u}+\frac{b_1a}{a_1(f+q)}\sqrt{\left(\frac{q-\bar{u}}{q-u}\right)}\frac{\bar{b}_1\bar{b}}{g+q}+\frac{c_1a}{a_1(f+r)}\sqrt{\left(\frac{r-\bar{u}}{r-u}\right)}\frac{\bar{c}_1\bar{b}}{g+r}$$

$$=\frac{a\bar{b}}{a_1}\left\{\frac{a_1^{\,2}}{\overline{f+u}\,\overline{g+u}}+\frac{b_1\bar{b}_1}{\overline{f+q}\,\overline{g+q}}\sqrt{\left(\frac{q-\bar{u}}{q-u}\right)}+\frac{c_1\bar{c}_1}{\overline{f+r}\,\overline{g+r}}\sqrt{\left(\frac{r-\bar{u}}{r-u}\right)}\right\}$$

$$=\frac{ab_1}{a_1}\left\{\frac{a_1^{\,2}}{(f+u)(g+u)}+\frac{b_1^{\,2}}{(f+q)(g+q)}+\frac{c_1^{\,2}}{(f+r)(g+r)}\right\}=0;$$

and similarly for the coefficient of $\overline{Z}$. The coefficient of $\overline{X}$ is in like manner shewn to be

$$\frac{a\bar{a}}{a_1}\left\{\frac{a_1^2}{(f+u)^2}+\frac{b_1^2}{(f+q)^2}+\frac{c_1^2}{(f+r)^2}\right\}$$

$$=\frac{a\bar{a}}{a_1}\,\frac{(f-g)(f-h)}{(f+u)(f+q)(f+r)}=\frac{a\bar{a}}{a^2a_1}=\frac{\bar{a}}{aa_1};$$

or the quantity in question is simply

$$\frac{\bar{a}}{aa_1}\,\overline{X}.$$

Multiplying this by $\overline{X}_1=\bar{a}\,\overline{X}+\bar{a}'\,\overline{Y}+\bar{a}''\overline{Z}$, the terms containing $\overline{X}\,\overline{Y}$, $\overline{X}\,\overline{Z}$ vanish after the integration, and we need only consider the term $\dfrac{\bar{a}\,\bar{a}}{aa_1}\,\overline{X}^2$, or what is the same $\dfrac{\bar{a}^2\bar{a}_1}{aa_1(f+\bar{u})}\,\overline{X}^2$. Whence

$$A=\sqrt{\left\{\frac{(\bar{u}-q)(\bar{u}-r)}{(u-q)(u-r)}\right\}}\;\frac{\bar{a}^2\bar{a}_1}{aa_1(f+\bar{u})}\int\frac{\overline{X}^2d\overline{S}}{\dfrac{\overline{X}^2}{f+u}+\dfrac{\overline{Y}^2}{g+u}+\dfrac{\overline{Z}^2}{h+u}}.$$

The value of the corresponding function $\overline{A}$ (*i.e.* the attraction of the exterior ellipsoid upon $\overline{P}$) is

$$\overline{A}=\frac{\bar{a}}{f+u}\int\frac{\overline{X}^2d\overline{S}}{\dfrac{\overline{X}^2}{f+u}+\dfrac{\overline{Y}^2}{g+u}+\dfrac{\overline{Z}^2}{h+u}},$$

the limits being the same, whence

$$\frac{A}{\overline{A}}=\sqrt{\left\{\frac{(\bar{u}-q)(\bar{u}-r)}{(u-q)(u-r)}\right\}}\;\frac{f+u}{f+\bar{u}}\;\frac{\bar{a}\,\bar{a}_1}{aa_1}=\frac{\sqrt{(\bar{u}+g)}\sqrt{(\bar{u}+h)}}{\sqrt{(u+g)}\sqrt{(u+h)}};$$

or we have

$$A=\frac{\sqrt{(\bar{u}+g)}\sqrt{(\bar{u}+h)}}{\sqrt{(u+g)}\sqrt{(u+h)}}\overline{A},\quad B=\frac{\sqrt{(\bar{u}+h)}\sqrt{(\bar{u}+f)}}{\sqrt{(u+h)}\sqrt{(u+f)}}\overline{B},\quad C=\frac{\sqrt{(\bar{u}+f)}\sqrt{(\bar{u}+g)}}{\sqrt{(u+f)}\sqrt{(u+g)}}\overline{C},$$

formulæ which constitute in fact Ivory's theorem.

Let K, $\overline{K}$ denote the attractions in the directions of the normals at P, $\overline{P}$, we have

$$K=\frac{\bar{\mu}a_1}{\mu a_1}\int\overline{X}_1d\overline{S},\quad \overline{K}=\int\overline{X}_1d\overline{S},$$

or
$$K = \frac{\overline{\mu a_1}}{\mu a_1} K$$
and it is important to remark that this is true not only for the entire ellipsoids; but if $\mathfrak{M}$, $\mathfrak{N}$ denote the attractions of the cones standing on the portions $\overline{M}$, $\overline{N}$ of the surface interior ellipsoid, and $\overline{\mathfrak{M}}$, $\overline{\mathfrak{N}}$ the attractions of the portions of the exterior ellipsoid bounded by the tangent plane at $\overline{P}$, and the portions M, N of the surface of the exterior ellipsoid, then
$$\mathfrak{M} = -\frac{\overline{\mu a_1}}{\mu a_1} \overline{\mathfrak{M}}, \quad \mathfrak{N} = \frac{\overline{\mu a_1}}{\mu a_1} \overline{\mathfrak{N}},$$
where obviously
$$K = \mathfrak{N} - \mathfrak{M}, \quad \overline{K} = \overline{\mathfrak{N}} + \overline{\mathfrak{M}},$$
this theorem is so far as I am aware new.

ON THE RELATIONS WHICH EXIST IN CERTAIN SYSTEMS OF POINTS AND PLANES, INCLUDING THE RELATION BETWEEN TEN POINTS IN A SURFACE OF THE SECOND DEGREE.

By Thomas Weddle.

Take six of ten points in a surface of the second degree for the angles of an octahedron, and denote the other four points by a_1, a_2, a_3, and a_4.

Let $t = 0$, $t' = 0$, $u = 0$, $u' = 0$, $v = 0$, $v' = 0$, $w = 0$, and $w' = 0$, be the equations to the faces of the octahedron, t and t' being opposite, &c.; also let t, t', u, u', &c. have been multiplied by such constants that if a straight line be drawn through *any* point (Q) parallel to some fixed line (taken at pleasure), the values which t, t', u, u', &c. take at Q shall be equal to the segments of the straight line intercepted between Q and the planes t, t', u, u', &c. respectively. Moreover let t_n, t_n', u_n, u_n', v_n, v_n', w_n, and w_n'* denote the portions of the straight line drawn through the point a_n parallel to the fixed line, and intercepted between a_n and the planes t, t', u, u', v, v', w, and w' respectively.

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* It is scarcely necessary to observe that lines lying on opposite sides of the same plane must be taken with different signs. Thus such of the lines t_1, t_2, t_3, t_4, as lie on one side (it matters not which) of the plane t being taken positively, those (if any) that lie on the other side must be taken negatively.

Since the surface of the second degree is circumscribed about the octahedron, its equation (*Journal*, vol. v., New Series, p. 66, foot-note), must be of the form

$$gtt' + huu' + kvv' + lww' = 0 \;\ldots\ldots\ldots (1),$$

where g, h, k, and l are constants.

Now the surface also passes through the points a_1, a_2, a_3, and a_4, and we must therefore have

$$gt_1t_1' + hu_1u_1' + kv_1v_1' + lw_1w_1' = 0 \;\ldots\ldots (2),$$
$$gt_2t_2' + hu_2u_2' + kv_2v_2' + lw_2w_2' = 0 \;\ldots\ldots (3),$$
$$gt_3t_3' + hu_3u_3' + kv_3v_3' + lw_3w_3' = 0 \;\ldots\ldots (4),$$

and

$$gt_4t_4' + hu_4u_4' + kv_4v_4' + lw_4w_4' = 0 \;\ldots\ldots (5).$$

Eliminate g, h, k, and l from these four equations, and we get an expression which may be put under various forms; one of which is

$$\begin{aligned}
&(t_4t_4'u_3u_3' - t_3t_3'u_4u_4')(v_1v_1'w_2w_2' - v_2v_2'w_1w_1') \\
&\quad + (t_3t_3'v_4v_4' - t_4t_4'v_3v_3')(u_1u_1'w_2w_2' - u_2u_2'w_1w_1') \\
&\quad + (t_4t_4'w_3w_3' - t_3t_3'w_4w_4')(u_1u_1'v_2v_2' - u_2u_2'v_1v_1') \\
&\quad + (u_4u_4'v_3v_3' - u_3u_3'v_4v_4')(t_1t_1'w_2w_2' - t_2t_2'w_1w_1') \\
&\quad + (u_3u_3'w_4w_4' - u_4u_4'w_3w_3')(t_1t_1'v_2v_2' - t_2t_2'v_1v_1') \\
&\quad + (v_4v_4'w_3w_3' - v_3v_3'w_4w_4')(t_1t_1'u_2u_2' - t_2t_2'u_1u_1') = 0 \ldots (6).
\end{aligned}$$

This is a relation, both necessary and sufficient, among ten points in a surface of the second degree. The expression is doubtless a somewhat complicated one; but I shall presently, by a slight artifice, present the relation in a simpler geometrical form.

I have supposed that all the lines t, u, v, w, are drawn parallel to each other, so that t_n, t_n', u_n, u_n', v_n, v_n', w_n, and w_n', are segments of the same line drawn from a_n, and are the portions of this line intercepted between a_n and the planes t, t', u, u', v, v', w, and w' respectively. It is easily seen, however, that all that is necessary to the truth of (6), is that the four lines limited by the same plane shall be parallel, it not being necessary that the lines limited by different planes shall be parallel: thus t_1, t_2, t_3, t_4, must be parallel to each other, so also must t_1', t_2', t_3', t_4', but it is not necessary that the former set be parallel to the latter. We may hence evidently draw all the lines perpendicular to the respective planes, if we please.

Let us now inquire what are the relations among nine points situated on the curve of intersection of two surfaces

of the second degree; that is, of nine points such that every surface of the second degree passing through eight of them shall also pass through the ninth?

Take six of the points for the angles of an octahedron, denote the other three points by a_1, a_2, a_3, and retain the previous notation. We shall here have the three conditions (2), (3), and (4); and since every surface of the second degree passing through eight of the points is to pass through the ninth, two of these equations must imply the third. Multiply therefore (2), (3), and (4) by λ, μ, and ν respectively; add the products and equate the coefficients of g, h, k, and l to zero, therefore

$$\left.\begin{aligned} \lambda t_1t_1' + \mu t_2t_2' + \nu t_3t_3' &= 0 \\ \lambda u_1u_1' + \mu u_2u_2' + \nu u_3u_3' &= 0 \\ \lambda v_1v_1' + \mu v_2v_2' + \nu v_3v_3' &= 0 \\ \lambda w_1w_1' + \mu w_2w_2' + \nu w_3w_3' &= 0 \end{aligned}\right\} \ldots\ldots.(7).$$

The elimination of λ, μ, and ν from these equations will give the two relations required that every surface of the second degree passing through eight of the points shall also pass through the ninth. The results of this elimination can of course be exhibited in various forms: one way will be got by eliminating λ, μ, ν; 1st, from the first three equations, 2nd, from the last three equations; therefore

$$\left.\begin{aligned} (u_1u_1'v_2v_2' - u_2u_2'v_1v_1')t_3t_3' + (v_1v_1't_2t_2' - v_2v_2't_1t_1')u_3u_3' & \\ + (t_1t_1'u_2u_2' - t_2t_2'u_1u_1')v_3v_3' &= 0 \\ (v_1v_1'w_2w_2' - v_2v_2'w_1w_1')u_3u_3' + (w_1w_1'u_2u_2' - w_2w_2'u_1u_1')v_3v_3' & \\ + (u_1u_1'v_2v_2' - u_2u_2'v_1v_1')w_3w_3' &= 0 \end{aligned}\right\} \ldots.(8).$$

These equations may be employed to construct the curve when eight points in it are given, but it will be more convenient to proceed a little differently, and I shall recur to the subject presently.

Again, let us seek the relations among the eight points in which three surfaces of the second degree intersect, that is, the relations among eight points such that every surface of the second degree passing through seven of them shall also pass through the eighth.

Take, as before, six of the eight points for the faces of an octahedron, and let a_1 and a_2 be the other two points. We shall here have the two conditions, (2) and (3), and in order that the eight points may be such a system as supposed, it is clear that these equations must be identical, and this requires

the three relations

$$\frac{t_2t_2'}{t_1t_1'}=\frac{u_2u_2'}{u_1u_1'}=\frac{v_2v_2'}{v_1v_1'}=\frac{w_2w_2'}{w_1w_1'}\ldots\ldots\ldots\ldots(9).$$

These expressions are remarkably simple, and they are perfectly analogous to the relation among six points in a conic, known as Desargues's 'Involution of six points'; indeed I consider each of the relations (6), (8), and (9) as analogous to Desargues's theorem.

It does not seem easy to apply (9) to find the eighth point when seven are given, but I have fallen upon a method of doing this founded on other principles. I shall add this construction at the end of the paper.

I have said that we may, by a slight artifice, present the relation among ten points in a surface of the second degree, in a geometrical form. This may be done as follows:

If we consider l to be a *line* of any given length, and g, h, k to be the variable coordinates of a point, it is clear that (2) will denote a plane, which is moreover easily constructible, for the intercepts on the axes of g, h, and k are

$$-\frac{lw_1w_1'}{t_1t_1'},\quad -\frac{lw_1w_1'}{u_1u_1'},\quad \text{and}\ -\frac{lw_1w_1'}{v_1v_1'}$$

respectively, and each of these can be found by a double application of Euclid VI. 12 (To find a fourth proportional to three given straight lines). Construct then in this way the four planes whose equations are (2), (3), (4), and (5), and we see that the condition, at once necessary and sufficient, that the ten points shall be in a surface of the second degree, is, that the four planes shall intersect in a point. Of course the coordinates of this point will be the values of g, h, and k.

Moreover, we may present the conditions required, in order that nine points may be situated on the curve of intersection of two surfaces of the second degree, as follows: Construct the three planes denoted by (2), (3), and (4); then if the nine points be such a system as supposed, the three planes will intersect in a straight line.

When nine points in a surface of the second degree are given, the preceding principles enable us to construct the surface. The points being the angles of an octahedron, and a_1, a_2, a_3, through a_3 draw any straight line, and let us seek the other point a_4 in which this line intersects the surface. Draw all the lines t, u, v, w, parallel to a_3a_4, and retain the previous notation. Assuming l equal to any line

at pleasure, construct the three planes whose equations are (2), (3), and (4), and find the coordinates g, h, and k* of their point of intersection, so that g, h, k, and l will now be all known. Let x be the distance between the points a_3 and a_4, then it is clear that

$$t_3-t_4=t_3'-t_4'=u_3-u_4=u_3'-u_4'=v_3-v_4=v_3'-v_4'=w_3-w_4=w_3'-w_4'=x;$$

also from (4) and (5),

$$g(t_3t_3'-t_4t_4')+h(u_3u_3'-u_4u_4')+k(v_3v_3'-v_4v_4')+l(w_3w_3'-w_4w_4')=0.$$

Eliminate t_4, t_4', $u_4 \ldots w_4'$ from this equation by means of the immediately preceding ones, and reduce; therefore

$$(g+h+k+l)x=g(t_3+t_3')+h(u_3+u_3')+k(v_3+v_3')+l(w_3+w_3') \ldots (10);$$

hence x is known, and thence the point a_4. Of course the surface will be constructed by finding in like manner the other point in which every straight line drawn through a_3 intersects the surface.

If, instead of finding the other point in which a line drawn through a_3 intersects the surface, we wish to find the *two* points in which *any* given straight line intersects it, we must draw all the t, u, v, and w parallel to this line, and find g, h, k, and l as above. The portions of the given line intercepted between the face t and the other faces t', u, u', v, v', w, w'

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* This construction may easily be effected in one plane. Having, by Euclid VI. 12, reduced (2), (3), and (4) to the forms

$$\frac{g}{a_1}+\frac{h}{b_1}+\frac{k}{c_1}=1,$$

$$\frac{g}{a_2}+\frac{h}{b_2}+\frac{k}{c_2}=1,$$

and $$\frac{g}{a_3}+\frac{h}{b_3}+\frac{k}{c_3}=1,$$

draw two straight lines OA_1 and B_1OC_1 perpendicular to each other; on OA_1 set off $OA_1=a_1$, $OA_2=a_2$, $OA_3=a_3$; on OB_1 set off $OB_1=b_1$, $OB_2=b_2$, $OB_3=b_3$; and on OC_1 set off $OC_1=c_1$, $OC_2=c_2$, $OC_3=c_3$; draw A_2B_2 and A_3B_3 intersecting A_1B_1 in D and E; join A_2C_2, and A_3C_3 intersecting A_1C_1 in F and G; draw FH and GI perpendicular to OA_1, and join DH and EI intersecting in K; draw KR perpendicular to OA_1, and, if DH and HF be not in a straight line, join DF intersecting KR in P; then shall $OR=g$, $RK=h$, and $KP=k$. If DH and HF should be in a straight line, from H draw any straight line $QH=HF$, and draw KS parallel to QH, meeting DQ in S; then shall $KS=k$.

This construction is immediately derived from Descriptive Geometry, the final step being slightly modified.

are of course known; let these be denoted by a', b, b', c, c', d, d' respectively; then will $t_4' = t_4 + a'$, $u_4 = t_4 + b$, $u_4' = t_4 + b'$, $v_4 = t_4 + c$, $v_4' = t_4 + c'$, $w_4 = t_4 + d$, and $w_4' = t_4 + d'$; these values being substituted in (5), will give the following quadratic to determine t_4,

$$(g + h + k + l)t_4^2 + \{ga' + h(b + b') + k(c + c') + l(d + d')\} t_4 + hbb' + kcc' + ldd' = 0 \ldots\ldots\ldots\ldots (11).$$

The two values of t_4 derived from this equation (either by algebraical solution or geometrical construction) will determine the two points in which the surface is intersected by the given line.

The latter method of constructing the surface has one advantage over the former. Each new point determined by the former method will require new values of g, h, k, and l to be found; while in the latter method, the values of these quantities require to be found once only, for we can construct the surface by finding the points in which parallel lines intersect it.

I next proceed to construct the curve of intersection of two surfaces of the second degree, when eight points in it are given.

Take, as usual, six of the eight points for the angles of an octahedron, and let α_1 and α_2 be the other two points. Through α_1 and α_2 draw any plane Z, and let us seek the other two points in which this plane intersects the curve. Draw all the lines t, u, v, w parallel to any line Y in the plane Z. The surfaces whose equations are

$$gtt' + huu' + kvv' = 0 \ldots\ldots\ldots\ldots (12),$$

and

$$h'uu' + k'vv' + l'ww' = 0 \ldots\ldots\ldots\ldots (13),$$

will each pass through all the eight points, providing

$$\left.\begin{aligned} gt_1t_1' + hu_1u_1' + kv_1v_1' &= 0 \\ gt_2t_2' + hu_2u_2' + kv_2v_2' &= 0 \end{aligned}\right\} \ldots\ldots\ldots (14),$$

and

$$\left.\begin{aligned} h'u_1u_1' + k'v_1v_1' + l'w_1w_1' &= 0 \\ h'u_2u_2' + k'v_2v_2' + l'w_2w_2' &= 0 \end{aligned}\right\} \ldots\ldots\ldots (15).$$

Hence the two points sought will be found by the intersection of the conics in which the surfaces (12) and (13) are cut by the plane Z. (For brevity I shall call these the conics (Z, 12) and (Z, 13)).

To find g, h, and k, assume k equal to any line at pleasure; construct the two straight lines, (g and h being the variable coordinates), whose equations are (14), and the

coordinates of their point of intersection will be g and h. Similarly h', k', and l' may be found.

Let the straight line drawn through a_1 parallel to Y intersect the conic $(Z, 12)$ in the point a and put $a_1a = x$, then at the point a we shall have $t = t_1 - x$, $t' = t_1' - x$, $u = u_1 - x$, $u' = u_1' - x$, $v = v_1 - x$, $v' = v_1' - x$, $w = w_1 - x$, and $w' = w_1' - x$. Now (12, 14),

$$g(t_1t_1' - tt') + h(u_1u_1' - uu') + k(v_1v_1' - vv') = 0;$$

eliminate t, t', &c. by the equations just given, and reduce, therefore

$$(g + h + k)x = g(t_1 + t_1') + h(u_1 + u_1') + k(v_1 + v_1');$$

hence x is known, and thence the point a.

Similarly we find the point d in which the conic $(Z, 13)$ is intersected by the line through a_1 parallel to Y; also the points b and e in which the line drawn through a_2 parallel to Y intersects the conics $(Z, 12)$ and $(Z, 13)$ respectively; join ab and de intersecting in p.

Again, instead of drawing all the lines t, u, v, w parallel to the line Y in the plane Z, let us draw them parallel to some other line X in that plane. Repeat the entire process (for new values of g, h, &c. will be required), and find the points c and f in which the conics $(Z, 12)$ and $(Z, 13)$ are intersected by the line drawn through a_1 parallel to X; join bc and ef intersecting in q. Now a little consideration will make it evident that since a_1 and a_2 are two of the points of intersection of the conics $(Z, 12)$ and $(Z, 13)$, the points p and q are both situated on the common chord joining the other two points of intersection; join therefore p and q, and find the points γ and δ* in which pq intersects either of the conics; then will γ and δ be the other two points of intersection of these conics, and therefore the other two points in which the plane Z intersects the curve to be constructed. By finding, in like manner, the points in which every plane drawn through a_1 and a_2 intersects the curve, the curve itself will be constructed.

Moreover, it is clear that if we can transform (6), (8), and (9) by the theory of reciprocal polars, we shall get relations among the reciprocal systems of ten, nine, and eight planes; and this transformation can be effected by aid

* Since five points a_1, a_2, a, b, c are given on the conic $(Z, 12)$, the points γ and δ in which this conic is intersected by pq may be found by known methods.

of the following principle given by M. Chasles in his *Aperçu Historique*, p. 688.

Two fixed points, a and b, being taken in space, and their polar planes A and B, with respect to a surface of the second degree, the ratio of the distances of any transversal plane whatever, from the two fixed points a and b, will be to the ratio of the distances of the pole of this plane from the two planes A and B in a constant ratio.

Hence if T_n, T_n', U_n, U_n', V_n, V_n', W_n, and W_n' denote the lines in the reciprocal system corresponding to the lines t_n, t_n', u_n, u_n', v_n, v_n', w_n, and w_n' respectively; and λ_n the corresponding constant, we shall have the following equations:

$$\frac{t_n}{t_1} = \lambda_n \frac{T_n}{T_1}, \quad \frac{t_n'}{t_1'} = \lambda_n \frac{T_n'}{T_1'}, \quad \frac{u_n}{u_1} = \lambda_n \frac{U_n}{U_1}, \quad \frac{u_n'}{u_1'} = \lambda_n \frac{U_n'}{U_1'},$$

$$\frac{v_n}{v_1} = \lambda_n \frac{V_n}{V_1}, \quad \frac{v_n'}{v_1'} = \lambda_n \frac{V_n'}{V_1'}, \quad \frac{w_n}{w_1} = \lambda_n \frac{W_n}{W_1}, \text{ and } \frac{w_n'}{w_1'} = \lambda_n \frac{W_n'}{W_1'}.$$

Eliminate all the t, u, v, w from (6), (8), and (9) by means of these equations, and reduce; we thus find relations of the very same forms, the only difference being that the transformed equations have T, U, V, and W instead of t, u, v, and w.

Hence, recollecting that the reciprocals of the angles and faces of an octahedron are the faces and angles of a hexahedron, it is easy to see the truth of the following relations.

1. Equation (6) expresses the relation among ten tangent planes to a surface of the second degree as follows:

Take six of the ten planes for the faces of a hexahedron, and let A_1, A_2, A_3, and A_4 be the other four planes. From the angles of the hexahedron draw parallel lines, and let t_n, t_n', u_n, u_n', v_n, v_n', w_n, and w_n', denote the portions of these lines intercepted between the angles of the hexahedron and the plane A_n, t_n and t_n' being drawn from opposite angles, &c. Then (6) is the relation, both necessary and sufficient, in order that the ten planes may touch a surface of the second degree.

It is easily seen that while the lines limited by the same plane must be parallel, those limited by different planes need not be so; thus the lines $t_1 t_1' u_1 u_1' \ldots w_1'$ limited by the plane A_1 must be parallel to each other; so also must $t_2 t_2' u_2 u_2' \ldots w_2'$ limited by the plane A_2; but it is not necessary that the former lines should be parallel to the latter. It thus appears that we may, if we please, draw all the lines perpendicular to the respective planes.

2. In like manner, equations (8) express the relations among nine tangent planes to a developable circumscribed about two surfaces of the second degree; or, which is the same thing, among nine planes which are such that every surface of the second degree touching eight of them shall also touch the ninth. And,

3. Equations (9) express the relations among the eight common tangent planes to three surfaces of the second degree, or, which is the same thing, among eight planes which are such that every surface of the second degree touching seven of them shall also touch the eighth.

Of course the relations may be expressed in the geometrical forms previously alluded to.*

It will now be evident that the problem proposed by the Brussels Academy,—'To find the relation among ten points in a surface of the second degree' has at length received a solution; and I do not doubt but other solutions will speedily follow, of a simpler character and capable probably of more extensive applications than mine.

Addendum. I now add the solution of the problem alluded to at p. 229.

Problem. Let A, B, C, D, E, F, G and H be the eight points in which three surfaces of the second degree intersect; that is, a system of eight points such that every surface of the second degree passing through seven of them also passes through the eighth. Given the first seven points, to find the eighth (H).

Let the straight line in which the planes ABC, EFG intersect, meet the planes BCD, CDE, and DEF in the points P, Q, and R, respectively; also let the straight lines FG, AG, and AB intersect the same planes in L, M, and N, respectively, all these points &c. being thus known. Suppose the planes BCD and FGH to intersect in the straight line SLT, the planes CDE and GHA in the straight line UMV, and the planes DEF and HAB in the straight line WNX. Also suppose T and U to be situated on DC the intersection of the planes BCD and CDE, W and V on DE that of the planes CDE and DEF, and S and X on DK that of the planes BCD and DEF. Now I have shewn in my second memoir on the 'Theorems in Space analogous to those of

* I have not entered into the question of construction in the case of planes, for we have only to take the reciprocal of the system with respect to any sphere, and apply the construction for the case of as many points.

Pascal and Brianchon in a Plane,'* that 'If eight planes intersect three and three in order in eight points such that every surface of the second degree passing through seven of them shall also pass through the eighth, the opposite planes will intersect in four straight lines, belonging to the same system of generators, in an hyperboloid of one sheet.' Hence the straight lines PQR, SLT, UMV, and WNX are generators (of the same species) of an hyperboloid. Consequently the three points P, U, X, in which the straight lines PQR, UMV, and WNX intersect the tangent plane BCD of the hyperboloid, must range in a straight line; that is, UX passes through P. Similarly TW passes through Q, and VS through R. Thus we have a twisted hexagon $STWXUV$, whose sides ST, TW, WX, XU, UV, VS pass through the given points L, Q, N, P, M, R, respectively, and whose angular points rest upon the straight lines DC, DE, and DK, the mutual intersections of the planes BCD, CDE, and DEF. But, 'If a variable polygon, either plane or twisted, have its angles on straight lines meeting in a point; and if, moreover, all its sides except one pass through fixed points, that one will also pass through a fixed point'; and this point can, of course, be found by constructing two particular polygons. We can therefore determine a second point through which any side of the hexagon must pass, and the following construction for the eighth point will now be sufficiently intelligible.

Having found the points P, Q, R, L, M, N, and the straight lines DC, DE, and DK, as before, take any point S' in DK, join $S'L$ intersecting DC in T'; join $T'Q$ intersecting DE in W', join $W'N$ intersecting DK in X', join $X'P$ intersecting DC in U', join $U'M$ intersecting DE in V'; finally, join $V'S'$, thus forming the hexagon $S'T'W'X'U'V'$. Assume any second point S'' in DK, and construct in like manner the hexagon $S''T''W''X''U''V''$; let $S'V'$ and $S''V''$ intersect in Y; through R and Y draw the straight line SV intersecting DK in S, and construct the hexagon $STWXUV$ in the same manner as $S'T'W'X'U'V'$ was constructed. Through the straight lines SLT and FGL draw a plane, through UMV and AGM draw a second plane, and through WNX and ABN draw a third plane; the three planes thus drawn are the planes FGH, GHA, and HAB, respectively, which will therefore determine the point H by their intersection.

Allen Villas, Battersea, May 28, 1850.

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* *Math. Journal*, New Series, vol. v., p. 58.

ON THE RELATION AMONG TEN POINTS IN A CURVE OF THE THIRD DEGREE.

By THOMAS WEDDLE.

TAKE six of the ten points for the angles of a hexagon, and let the other points be denoted by a_1, a_2, a_3, and a_4.

Let $t=0$, $u'=0$, $v=0$, $t'=0$, $u=0$, and $v'=0$, be the equations to the sides of the hexagon taken in order. Also let $T=0$, $U=0$, and $V=0$, be the equations to the diagonals joining opposite angles, T joining the points (uv') and $(u'v)$, &c. Suppose, moreover, that t, u, v, t', u', v', T, U, V, have been multiplied by such constants that if a straight line be drawn through *any* point (Q) parallel to some fixed line (taken at pleasure), the values which t, u, v, &c. take at Q shall be equal to the segments of the straight line intercepted between Q and the straight lines t, u, v, &c. respectively. Also let t_n, u_n, v_n, t_n', u_n', v_n', T_n, U_n, V_n denote the portions of the straight line drawn through a_n parallel to the fixed line, and intercepted between a_n and the straight lines t, u, v, t', u', v', T, U, V, respectively.

Since the curve of the third degree is circumscribed about the hexagon, it is clear that its equation may be denoted by

$$\lambda tt'T + \mu uu'U + \nu vv'V + \rho TUV^* = 0 \ldots\ldots (1),$$

where λ, μ, ν, and ρ are constants.

Since this curve also passes through the points a_1, a_2, a_3, and a_4, we shall have

$$\lambda t_1t_1'T_1 + \mu u_1u_1'U_1 + \nu v_1v_1'V_1 + \rho T_1U_1V_1 = 0\ldots.(2),$$

$$\lambda t_2t_2'T_2 + \mu u_2u_2'U_2 + \nu v_2v_2'V_2 + \rho T_2U_2V_2 = 0\ldots.(3),$$

$$\lambda t_3t_3'T_3 + \mu u_3u_3'U_3 + \nu v_3v_3'V_3 + \rho T_3U_3V_3 = 0\ldots.(4),$$

and $$\lambda t_4t_4'T_4 + \mu u_4u_4'U_4 + \nu v_4v_4'V_4 + \rho T_4U_4V_4 = 0\ldots.(5).$$

Hence if λ, μ, ν, and ρ be eliminated from these equations, we shall have the relation among ten points in a curve of the third degree. The result is so complicated that I shall not put it down: it seems nevertheless to be analogous to Desargues's involution of six points.

The preceding expressions enable us to construct the curve when nine points in it are given. For, take six of the points for the angles of a hexagon, and let a_1, a_2, a_3 be the other three points, and let us retain the preceding notation. Through a_3 draw any line, and let us seek the other two

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* Instead of ρTUV, we might evidently write either ρtuv or $\rho t'u'v'$.

points (let a_4 denote either of them) in which this straight line intersects the curve, and draw all the lines $t, u, v, t', u', v', T, U, V$ parallel to this straight line.

The equations (2, 3, 4) will determine λ, μ, ν, ρ (assuming one of them equal to any line at pleasure) either by algebraic solution or geometrical construction. Were it legitimate to make use of a construction in space to solve a plane problem, we should only have to assume ρ equal to any line at pleasure; construct the planes of which (2), (3) and (4) are the equations (λ, μ, and ν being the variable coordinates), and find the coordinates λ, μ, ν of their point of intersection: but this seems scarcely allowable. Perhaps even the construction in one plane, alluded to in a foot-note to the preceding paper, may be objected to, since it has been derived by aid of Descriptive Geometry; if so, we may proceed as follows:

By finding a number of fourth proportionals, we can reduce (2), (3), and (4) to the following,

$$\left.\begin{array}{l}\lambda a + \mu b \ + \nu c \ + \rho d \ = 0\\ \lambda a + \mu b' + \nu c' + \rho d' = 0\\ \lambda a + \mu b'' + \nu c'' + \rho d'' = 0\end{array}\right\}\ldots\ldots\ldots\ldots(6),$$

where all the a, b, c, d are lines. Therefore

$$\left.\begin{array}{l}\mu(b-b') + \nu(c-c') + \rho(d-d') = 0\\ \text{and}\quad \mu(b-b'') + \nu(c-c'') + \rho(d-d'') = 0\end{array}\right\}\ldots\ldots(7).$$

Assume ρ at pleasure, construct the two straight lines whose equations are (7), and find the coordinates μ, ν of their point of intersection; μ, ν and ρ being thus known, λ will be determined from any one of the equations (6), by finding three fourth proportionals.

Let x denote the distance between a_3 and a_4; then evidently $t_3-t_4=u_3-u_4=v_3-v_4=t_3'-t_4'=u_3'-u_4'=v_3'-v_4'=T_3-T_4=U_3-U_4=V_3-V_4=x$; deduct (4) from (5), and eliminate $t_4, u_4, v_4, t_4', u_4', v_4', T_4, U_4, V_4$ from the result by means of the preceding equations; we thus get, after reduction, the following equation,

$$\begin{aligned}(\lambda+\mu+\nu+\rho)\,x^2 - \{\lambda\,(t_3+t_3'+T_3) + \mu\,(u_3+u_3'+U_3)&\\ +\nu(v_3+v_3'+V_3)+\rho\,(T_3+U_3+V_3)\}\ x&\\ +\lambda\,(t_3t_3'+t_3T_3+t_3'T_3)+\mu\,(u_3u_3'+u_3U_3+u_3'U_3)+\nu(v_3v_3'+v_3V_3+v_3'V_3)&\\ +\rho\,(T_3U_3+T_3V_3+U_3V_3)=0\ldots.(8).&\end{aligned}$$

The two values of x being found from this equation, we immediately know the two points in which the line drawn through α_3 intersects the curve; which will therefore be constructed by giving every possible position to this line through α_3.

Allen Villas, Battersea, May 29, 1850.

ON THE THEOREMS IN SPACE ANALOGOUS TO TWO PROPERTIES OF THE PLANE QUADRILATERAL.

By THOMAS WEDDLE.

THE properties alluded to are the following:

(A) *If three of the sides of a variable quadrilateral inscribed in a fixed conic always pass through three fixed points in a straight line, then also the fourth side will always pass through a fixed point in the same straight line;*

and the reciprocal theorem,

(B) *If three of the angles of a variable quadrilateral circumscribed about a fixed conic are always situated on three fixed straight lines that pass through a point, the fourth angle will be always situated on a straight line passing through the same point.*

In Space we have the following analogous theorems:

(a) *If five of the faces of a variable hexahedron inscribed in a fixed surface of the second degree always pass through five fixed lines in one plane, then also the sixth face will always pass through a fixed line in the same plane;*

and the reciprocal theorem,

(b) *If five of the angular points of a variable octahedron circumscribed about a fixed surface of the second degree are always situated on five fixed straight lines which pass through one point, the sixth angular point will be always situated on a fixed straight line passing through the same point.*

I proceed to establish (a).

Let $w = 0$...(1) denote the equation to the plane in which the five fixed lines are situated; also let

$$t = 0\ldots(2),\qquad u = 0\ldots(3),\qquad v = 0\ldots(4),$$

be the equations to three contiguous faces of the hexahedron

in any *given* position, and

$$t' = at + bu + cv + ew = 0 \ldots\ldots\ldots\ldots(5),$$
$$u' = a_1t + b_1u + c_1v + e_1w = 0 \ldots\ldots\ldots\ldots(6),$$
$$v' = a_2t + b_2u + c_2v + e_2w = 0 \ldots\ldots\ldots\ldots(7),$$

the equations to the opposite faces. Supposing t', u', and v' to have been multiplied by the proper constants, the equation to the fixed surface circumscribed about the hexahedron may be denoted by

$$tt' + uu' + vv' = 0,$$

or $$at^2 + b_1u^2 + c_2v^2 + (b + a_1)\,tu + (c + a_2)\,tv + (c_1 + b_2)\,uv$$
$$+ \text{ terms containing } w = 0.\ldots(8).$$

Now supposing the fixed lines to be those in which the plane (1) is intersected by the planes (2), (3), (4), (5) and (6), the equations to five of the faces of the *variable* hexahedron will be

$$T = t + \lambda w = 0, \quad U = u + \mu w = 0, \quad V = v + \nu w = 0,$$
$$T' = at + bu + cv + \rho w = 0,$$
and $$U' = a_1t + b_1u + c_1v + \sigma w = 0.$$

Hence, taking the equation to the sixth face to be

$$V' = \alpha t + \beta u + \gamma v + \delta w = 0.\ldots\ldots\ldots\ldots(9),$$

(so that V' may be supposed multiplied by an arbitrary constant), we may denote the equation to the fixed surface circumscribed about the variable hexahedron by

$$TT' + \varepsilon UU' + VV' = 0,$$

which may be written

$$at^2 + \varepsilon b_1u^2 + \gamma v^2 + (b + \varepsilon a_1)\,tu + (c + \alpha)\,tv + (\varepsilon c_1 + \beta)\,uv$$
$$+ \text{ terms containing } w = 0.\ldots(10).$$

Now since (8) and (10) denote the very same surface, we must have

$$\varepsilon = 1, \quad \gamma = c_2, \quad \alpha = a_2, \quad \text{and } \beta = b_2;$$

hence the equation to the sixth variable face (9) becomes

$$a_2t + b_2u + c_2v + \delta w = 0,$$

which always passes through the fixed straight line

$$a_2t + b_2u + c_2v = w = 0,$$

in the plane (1). Hence (a) is established.

Again, the reciprocal of the angles and faces of an octahedron circumscribed about a surface of the second degree, will evidently be the faces and angles of a hexahedron inscribed in a surface of the second degree; also, if a point move on a straight line, its polar plane will turn round a straight line. Hence the reciprocal of the system mentioned in (b) will be a variable hexahedron inscribed in a fixed surface of the second degree, five of the faces of the hexahedron always passing through five fixed lines in one plane; hence, (a), the sixth face will always pass through a fixed line in the said plane; and consequently the polar of this face, that is the sixth angular point of the octahedron, will always be situated in a fixed line passing through the point in which the other five fixed lines intersect.

It will of course be understood that I do not present the theorems (a) and (b), (which so far as I know are now published for the first time), as the *only* ones that may be considered analogous to (A) and (B). Indeed Mr. Finlay, in his ingenious 'Application of Algebra to the Modern Geometry,' has (*Mathematician*, vol. II. p. 144) given the following theorem (α) as an analogue to (A).

(α) *If three of four variable planes which intersect two and two in order in straight lines in a fixed surface of the second degree, always pass through three fixed points in a straight line, then also the fourth plane will always pass through a fixed point in the same straight line.**

Also if we transform this theorem by the theory of reciprocal polars, we get the following analogue to (B).

(β) *If four variable planes intersect two and two in order in straight lines in a fixed surface of the second degree, and if three of the four points in which the said planes intersect three and three in order be always situated in three planes that intersect in a straight line, then also the fourth point will be always situated in a plane passing through the same straight line.*

The following demonstration of (α) may not perhaps be unacceptable to some readers.

Let $\qquad v = 0 \ldots (11), \quad w = 0 \ldots (12),$

denote any two fixed planes passing through the line in

* Of course the planes, points, &c. mentioned in (α) and (β), will not be real unless the surface be rule or developable; and therefore these theorems cannot, geometrically speaking, be considered properties of the umbilical surfaces, (that is, the ellipsoid, hyperboloid of two sheets, and elliptic paraboloid).

which the three fixed points are situated; also let

$$t = 0\ldots(13), \qquad u = 0\ldots(14),$$

$$v' = at + bu + cv + ew = 0 \;\ldots\ldots\ldots\; (15),$$

and
$$w' = ft + gu + hv + kw = 0 \;\ldots\ldots\ldots\; (16),$$

be the equations to the four planes in any given position; then supposing w' to have been multiplied by the proper constant, we may denote the equation to the surface by

$$tv' + uw' = 0,$$

or $at^2 + gu^2 + (b+f)\,tu +$ terms containing v and $w = 0\ldots(17)$.
Again, supposing the three fixed points to be those in which the straight line (11, 12) is intersected by the planes (13), (14), and (15) respectively, the four *variable* planes may be denoted by

$$T = t + \lambda v + \lambda' w = 0,$$

$$U = u + \mu v + \mu' w = 0,$$

$$V = at + bu + \nu v + \nu' w = 0,$$

and
$$W = f't + g'u + h'v + k'w = 0 \;\ldots\ldots\ldots\; (18).$$

Also, since W may be supposed multiplied by an arbitrary constant, the fixed surface may be denoted by

$$TV + UW = 0,$$

or $at^2 + g'u^2 + (b+f')\,tu +$ terms containing v and $w = 0\ldots(19)$.

Now (17) and (19) denote the very same surface, and we must therefore have

$$g' = g, \text{ and } f' = f;$$

hence (18), the equation to the fourth variable plane becomes $ft + gu + h'v + k'w = 0$; and since this is always satisfied by $v = w = ft + gu = 0$, we infer that the fourth variable plane always passes through a fixed point on the line (11, 12).*

We may here notice that the theorems (α) and (β) are not essentially different; for the four planes referred to in (α) are evidently tangent planes, and the points in which they intersect three and three in order are their points of contact: now if a tangent plane always pass through a point, its point of contact will be always situated in the polar plane of that point; consequently, since the four planes always pass through as many points in a straight line, the four points in

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* If in the above investigation we consider $t=0$, $u=0$, $v=0$, $v'=0$, $w'=0$, $T=0$, $U=0$, $V=0$, and $W=0$, to denote straight lines, and reject w wherever it may occur, we shall obtain a demonstration of (A).

which they intersect three and three in order will always be situated in four planes that intersect in a straight line. It is scarcely necessary to observe that these four points are of course restricted to the conics in which the surface is intersected by the four planes.

It is to be remarked that as the theorems (A) and (B) admit of generalization (by writing $2n-1$ for *three*, *polygon of* $2n$ *sides* for *quadrilateral*, and $2n^{\text{th}}$ *side* or *angle* for *fourth side* or *angle*), so in like manner (α) and (β) may be generalized as follows:*

(*a*) *If* $2n-1$ *of* $2n$ *variable planes which intersect two and two in order in straight lines in a fixed surface of the second degree, always pass through* $2n-1$ *fixed points in a straight line, then also the* $2n^{\text{th}}$ *plane will always pass through a fixed point in the same straight line;* and

(*b*) *If* $2n$ *variable planes intersect two and two in order in straight lines in a fixed surface of the second degree, and if* $2n-1$ *of the* $2n$ *points in which the said planes intersect three and three in order be always situated in* $2n-1$ *planes that intersect in a straight line, then also the* $2n^{\text{th}}$ *point will be always situated in a plane passing through the same straight line.*

It will be sufficient to establish the former of these propositions, the latter being merely the reciprocal theorem.

Let the straight lines in which the planes intersect two and two in order, be denoted by $P_1, P_2, P_3 \dots P_{2n}$, the planes themselves being represented by

$$(P_1P_2),\ (P_2P_3), \dots\ (P_{2n-1}P_{2n}),\ (P_{2n}P_1),$$

the last being the '$2n^{\text{th}}$ plane' referred to. It is evident that if the fixed surface of the second degree be a twisted (gauche) surface, the straight lines $P_1, P_3, P_5, \dots P_{2n-1}$, will belong to one system of rectilinear generators, and $P_2, P_4, P_6 \dots P_{2n}$ to the other; while if the surface be developable, all these lines will meet in a point; so that in either case each of the first set of lines will be in a plane with each of the other set. Since the planes (P_1P_2), (P_2P_3), (P_3P_4), and (P_4P_1) intersect two and two in order in four straight lines in a surface of the second degree, and the first three planes always pass through three fixed points in a straight line, the fourth plane (P_4P_1) will, by (α), always pass through a fixed point in the same straight line. For a similar reason, since the three planes

* (a) and (b) do not seem to admit of so easy an extension.

(P_1P_4), (P_4P_5), and (P_5P_6) always pass through three fixed points in a straight line, the fourth plane (P_6P_1) will always pass through a fixed point in the same straight line. Continuing in this manner we may shew successively that the planes (P_1P_8), (P_1P_{10}),... (P_1P_{2n}) pass through fixed points in the straight line in which are situated the $2n-1$ fixed points, through which the $2n-1$ variable planes always pass.

Allen Villas, Battersea, Jan. 16, 1850.

PROPOSITIONS IN THE THEORY OF NUMBERS.

By the Rev. ARTHUR THACKER, M.A., Fellow of Trinity College, Cambridge.

1. LET the sum of the natural numbers from 1 to r be denoted by Σr^n, and the coefficient of x^p in the expansion of $(1+x)^n$ by n_p. The coefficient of x^n in

$$e^x + e^{2x} + e^{3x} + \dots + e^{rx}$$

is obviously $\dfrac{\Sigma r^n}{\lfloor\underline{n}}$. Now

$$e^x + e^{2x} + e^{3x} + \dots + e^{rx} = \frac{e^{(r+1)x} - e^x}{e^x - 1} = \frac{\{1 + (e^x - 1)\}^{r+1} - e^x}{e^x - 1}$$

$$= -1 + (r+1)_1 + (r+1)_2(e^x-1) + (r+1)_3(e^x-1)^2 + \dots + (r+1)_{p+1}(e^x-1)^p + \dots$$

$$= r + (r+1)_2(e^x-1) + (r+1)_3(e^{2x}-2e^x+1) + (r+1)_4(e^{3x}-3e^{2x}+3e^x-1) + \dots$$

$$+ (r+1)_{p+1}\{e^{px} - p_1 e^{(p-1)x} + \dots \pm 1\} + \dots,$$

whence, equating the coefficients of x^n, we obtain

$$\left.\begin{aligned} \Sigma r^n = (r+1)_2 &+ (2^n - 2.1^n)(r+1)_3 + (3^n - 3.2^n + 3.1^n)(r+1)_4 + \dots \\ &+ \{p^n - p_1(p-1)^n + p_2(p-2)^n - \dots \pm p\}(r+1)_{p+1} \\ &+ \dots \\ &+ \{n^n - n_1(n-1)^n + n_2(n-2)^n - \dots \pm n\}(r+1)_{n+1} \end{aligned}\right\} \dots (1).$$

The common expressions

$$\Sigma r = \frac{r(r+1)}{2}, \quad \Sigma r^2 = \frac{r(r+1)(2r+1)}{6}, \quad \Sigma r^3 = \left\{\frac{r(r+1)}{2}\right\}^2,$$

follow immediately from this series.

Substituting $a+m-1$ and $a-1$ successively for r, and

subtracting, we have

$$\left.\begin{aligned} a^n+(a+1)^n+\ldots+(a+m-1)^n &= (a+m)_2-a_2+(2^n-2)\{(a+m)_3-a_3\}+\ldots \\ &\quad+\{p^n-p_1(p-1)^n+\ldots\pm p\}\{(a+m)_{p+1}-a_{p+1}\} \\ &\quad+\ldots \\ &\quad+\{n^n-n_1(n-1)^n+\ldots\pm n\}\{(a+m)_{n+1}-a_{n+1}\} \end{aligned}\right\}\ldots(2),$$

which, allowing for a slight difference in notation, agrees with the expression obtained by a different and much longer method in Liouville's *Journal*, vol. XI. p. 487.

2. Another series for Σr^n, arranged in powers of r, may be obtained as follows:

$$\frac{e^{(r+1)x}-e^x}{e^x-1} = (e^{rx}-1)\frac{e^x}{e^x-1} = \frac{e^{rx}-1}{x}\cdot\frac{-x}{e^{-x}-1}.$$

Now $$\frac{x}{e^x-1} = 1-\frac{x}{2}+B_1\frac{x^2}{\lfloor\underline{2}}-B_3\frac{x^4}{\lfloor\underline{4}}+\ldots,$$

$B_1, B_3, \ldots$ being Bernouilli's numbers; whence

$$\frac{-x}{e^{-x}-1} = 1+\frac{x}{2}+B_1\frac{x^2}{\lfloor\underline{2}}-B_3\frac{x^4}{\lfloor\underline{4}}+\ldots$$

Also $$\frac{e^{rx}-1}{x} = r+\frac{r^2x}{\lfloor\underline{2}}+\frac{r^3x^2}{\lfloor\underline{3}}+\ldots+\frac{r^{n+1}x^n}{\lfloor\underline{n+1}}+\ldots$$

Collecting the coefficient of x^n in the product of these two series, we get

$$\frac{\Sigma r^n}{\lfloor\underline{n}} = \frac{r^{n+1}}{\lfloor\underline{n+1}}+\tfrac{1}{2}\cdot\frac{r^n}{\lfloor\underline{n}}+\frac{B_1}{\lfloor\underline{2}}\cdot\frac{r^{n-1}}{\lfloor\underline{n-1}}-\frac{B_3}{\lfloor\underline{3}}\cdot\frac{r^{n-3}}{\lfloor\underline{n-3}}+\ldots,$$

and therefore

$$\Sigma r^n = \frac{r^{n+1}}{n+1}+\tfrac{1}{2}.r^n+\tfrac{1}{2}n_1B_1r^{n-1}-\tfrac{1}{4}n_3B_3r^{n-3}+\tfrac{1}{6}n_5B_5r^{n-5}-\ldots.(3),$$

the last term being $(-1)^{\frac{1}{2}n+1}B_{n-1}r$, or $(-1)^{\frac{1}{2}(n+1)}\frac{1}{2}nB_{n-2}r^2$ according as n is even or odd.*

3. By comparing series (1) and (3), the values of Bernouilli's numbers may be found. For, from (1),

$$\begin{aligned}\Sigma r^{2n} = (r+1)_2+(2^{2n}-2)(r+1)_3+(3^{2n}-3.2^{2n}+3)(r+1)_4+\ldots \\ +\{(2n)^{2n}-(2n)_1(2n-1)^{2n}+\ldots\pm 2n\}(r+1)_{2n+1},\end{aligned}$$

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* For another proof, see *Crelle's Journal*, vol. XXXI. p. 252.

and from (3)

$$\Sigma r^{2n} = \frac{r^{2n+1}}{2n+1} + \tfrac{1}{2}r^{2n} + \tfrac{1}{2}(2n)_1 B_1 r^{2n-1} - \ldots + (-1)^{n+1} B_{2n-1} r.$$

Equating the coefficients of r, we have

$$\begin{aligned}(-1)^{n+1}B_{2n-1} = {} & \frac{1}{1.2} - (2^{2n}-2)\frac{1}{2.3} + (3^{2n}-3.2^{2n}+3)\frac{1}{3.4} - \ldots \\ & + (-1)^{2n-2}\{(2n)^{2n} - (2n)_1(2n-1)^{2n} + \ldots \pm 2n\}\frac{1}{2n(2n+1)} \\ = {} & \tfrac{1}{2} - \tfrac{1}{3}(2^{2n-1}-1) + \tfrac{1}{4}(3^{2n-1} - 2.2^{2n-1} + 1) - \ldots \\ & + \frac{(-1)^{2n-2}}{2n+1}\{(2n)^{2n-1} - (2n-1)_1(2n-1)^{2n-1} + \ldots \pm 1\}.\end{aligned}$$

4. By means of the series (3) we shall be able to prove the following theorem:

If N be an integer of the form $a^{\alpha}b^{\beta}c^{\gamma}\ldots$, $a, b, c, \ldots$ being prime factors, the sum of the n^{th} powers of the numbers which are less than N and prime to it

$$\begin{aligned}= \frac{N^{n+1}}{n+1}\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right)\ldots & + \tfrac{1}{2}n_1 B_1 N^{n-1}(1-a)(1-b)(1-c)\ldots \\ & - \tfrac{1}{4}n_3 B_3 N^{n-3}(1-a^3)(1-b^3)(1-c^3)\ldots \\ & + \tfrac{1}{6}n_5 B_5 N^{n-5}(1-a^5)(1-b^5)(1-c^5)\ldots \\ & - \ldots\end{aligned}$$

We have $\Sigma N^n = 1^n + 2^n + 3^n + \ldots + N^n$. Now if from this series we first subtract all the terms which are multiples of a, then from the remainder all the terms which are multiples of b, and again from the last remainder all those which are multiples of c, and so on; there will at last remain a series of terms, all of which are prime to N. This will be the sum required.

In the first place, the multiples of a between 1 and N are included in the series $a, 2a, 3a, \ldots \frac{N}{a}.a$, and the sum of the n^{th} powers of these

$$\begin{aligned}&= a^n + (2a)^n + (3a)^n + \ldots + \left(\frac{N}{a}.a\right)^n, \\ &= a^n\left\{1^n + 2^n + 3^n + \ldots + \left(\frac{N}{a}\right)^n\right\}, \\ &= a^n\Sigma\left(\frac{N}{a}\right)^n.\end{aligned}$$

Subtracting this from ΣN^n, and calling the difference R, we have

$$R = \Sigma N^n - a^n \Sigma \left(\frac{N}{a}\right)^n.$$

Again ΣN^n and $a^n \Sigma \left(\frac{N}{a}\right)^n$ both contain terms which are multiples of b; the sum of those in ΣN^n

$$= b^n + (2b)^n + (3b)^n + \ldots + \left(\frac{N}{b} . b\right)^n,$$

$$= b^n \Sigma \left(\frac{N}{b}\right)^n,$$

and the corresponding sum in $a^n \Sigma \left(\frac{N}{a}\right)^n$,

$$= a^n \left\{b^n + (2b)^n = (3b)^n + \ldots + \left(\frac{N}{ab} . b\right)^n\right\},$$

$$= a^n b^n \Sigma \left(\frac{N}{ab}\right)^n.$$

It follows that the total sum in

$$\Sigma N^n - a^n \Sigma \left(\frac{N}{a}\right)^n \text{ is } b^n \Sigma \left(\frac{N}{b}\right)^n - a^n b^n \Sigma \left(\frac{N}{ab}\right)^n,$$

which is the quantity to be subtracted from R. Calling the remainder R', we have

$$R' = \Sigma N^n - a^n \Sigma \left(\frac{N}{a}\right)^n - b^n \Sigma \left(\frac{N}{b}\right)^n + a^n b^n \Sigma \left(\frac{N}{ab}\right)^n.$$

In the same way, it is manifest that the sum of the terms in R', which are multiples of c,

$$= c^n \Sigma \left(\frac{N}{c}\right)^n - a^n c^n \Sigma \left(\frac{N}{ac}\right)^n - b^n c^n \Sigma \left(\frac{N}{bc}\right)^n + a^n b^n c^n \Sigma \left(\frac{N}{abc}\right)^n;$$

whence, subtracting this from R', and calling the remainder R'', we obtain

$$R'' = \Sigma N^n - a^n \Sigma \left(\frac{N}{a}\right)^n - b^n \Sigma \left(\frac{N}{b}\right)^n - c^n \Sigma \left(\frac{N}{c}\right)^n$$

$$+ b^n c^n \Sigma \left(\frac{N}{bc}\right)^n + a^n c^n \Sigma \left(\frac{N}{ac}\right)^n + a^n b^n \Sigma \left(\frac{N}{ab}\right)^n - a^n b^n c^n \Sigma \left(\frac{N}{abc}\right)^n.$$

In this expression, the law of formation is obvious. Supposing, for the sake of simplicity, that N contains three

prime factors only, R'' will be the sum required. We will denote this by σ_n, so that

$$\sigma_n = \Sigma N^n - a^n\Sigma\left(\frac{N}{a}\right)^n - b^n\Sigma\left(\frac{N}{b}\right)^n - c^n\Sigma\left(\frac{N}{c}\right)^n$$

$$+ b^n c^n\Sigma\left(\frac{N}{bc}\right)^n + a^n c^n\Sigma\left(\frac{N}{ac}\right)^n + a^n b^n\Sigma\left(\frac{N}{ab}\right)^n - a^n b^n c^n\Sigma\left(\frac{N}{abc}\right)^n.$$

Now by series (3), we have

$$\Sigma r^n = \frac{r^{n+1}}{n+1} + \tfrac{1}{2}r^n + \tfrac{1}{2}n_1B_1r^{n-1} - \tfrac{1}{4}n_3B_3r^{n-3} + \tfrac{1}{6}n_5B_5r^{n-5} - \ldots,$$

whence, substituting for ΣN^n, $\Sigma\left(\frac{N}{a}\right)^n \ldots$, we obtain

$$\begin{aligned}
\sigma_n = {} & \frac{N^{n+1}}{n+1} + \tfrac{1}{2}N^n + \tfrac{1}{2}n_1B_1N^{n-1} - \ldots. \\
& - a^n\left\{\frac{1}{n+1}\left(\frac{N}{a}\right)^{n+1} + \tfrac{1}{2}\left(\frac{N}{a}\right)^n + \tfrac{1}{2}n_1B_1\left(\frac{N}{a}\right)^{n-1} - \ldots.\right\} \\
& - \&c. \\
& + b^n c^n\left\{\frac{1}{n+1}\left(\frac{N}{bc}\right)^{n+1} + \tfrac{1}{2}\left(\frac{N}{bc}\right)^n + \tfrac{1}{2}n_1B_1\left(\frac{N}{bc}\right)^{n-1} - \ldots.\right\} \\
& + \&c. \\
& - a^n b^n c^n\left\{\frac{1}{n+1}\left(\frac{N}{abc}\right)^{n+1} + \tfrac{1}{2}\left(\frac{N}{abc}\right)^n + \tfrac{1}{2}n_1B_1\left(\frac{N}{abc}\right)^{n-1} - \ldots\right\} \\
= {} & \frac{N^{n+1}}{n+1}\left(1 - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} + \frac{1}{bc} + \frac{1}{ac} + \frac{1}{ab} - \frac{1}{abc}\right) \\
& + \tfrac{1}{2}N^n(1 - 1 - 1 - 1 + 1 + 1 + 1 - 1) \\
& + \tfrac{1}{2}n_1B_1N^{n-1}(1 - a - b - c + bc + ac + ab - abc) \\
& - \tfrac{1}{4}n_3B_3N^{n-3}(1 - a^3 - b^3 - c^3 + b^3c^3 + a^3c^3 + a^3b^3 - a^3b^3c^3) \\
& + \ldots\ldots\ldots\ldots\ldots \\
= {} & \frac{N^{n+1}}{n+1}\left(1-\frac{1}{a}\right)\left(1-\frac{1}{b}\right)\left(1-\frac{1}{c}\right) + \tfrac{1}{2}n_1B_1N^{n-1}(1-a)(1-b)(1-c) \\
& - \tfrac{1}{4}n_3B_3N^{n-3}(1-a^3)(1-b^3)(1-c^3) \\
& + \tfrac{1}{6}n_5B_5N^{n-5}(1-a^5)(1-b^5)(1-c^5) \\
& - \ldots\ldots\ldots\ldots
\end{aligned}$$

which is the expression required, the number of terms being $\frac{n+1}{2}$ or $\frac{n}{2}+1$, according as n is odd or even.

5. If we make $n = 0$, we get

$$\sigma_0 = N\left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{b}\right)\left(1 - \frac{1}{c}\right),$$

the known expression for the *number* of integers which are less than N and prime to it.

If $n = 1$, we have

$$\sigma_1 = \frac{N^2}{2}\left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{b}\right)\left(1 - \frac{1}{c}\right),$$

which gives the *sum* of the integers less than N and prime to it.

Also, since $B_1 = \frac{1}{6}$, we have

$$\sigma_2 = \frac{N^3}{3}\left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{b}\right)\left(1 - \frac{1}{c}\right) + \frac{N}{6}(1-a)(1-b)(1-c),$$

for the sum of the squares, and

$$\sigma_3 = \frac{N^4}{4}\left(1 - \frac{1}{a}\right)\left(1 - \frac{1}{b}\right)\left(1 - \frac{1}{c}\right) + \frac{N^2}{4}(1-a)(1-b)(1-c),$$

for the sum of the cubes.

Trinity College, Jan. 15, 1850.

THEORETICAL CONSIDERATIONS ON THE EFFECT OF PRESSURE IN LOWERING THE FREEZING POINT OF WATER.

By James Thomson.

(*Taken, with some slight alterations made by the author, from the Transactions of the Royal Society of Edinburgh,* vol. xvi. part v. 1849.)

Some time ago my brother (Professor William Thomson) pointed out to me a curious conclusion to which he had been led, by reasoning on principles similar to those developed by Carnot, with reference to the motive power of heat. It was, that *water at the freezing point may be converted into ice by a process solely mechanical, and yet without the final expenditure of any mechanical work.* This at first appeared to me to involve an impossibility, because water expands while freezing; and therefore it seemed to follow, that if a quantity of it were merely enclosed in a vessel with a moveable piston and frozen, the motion of the piston, consequent on the expansion, being resisted by pressure, mechanical work would

be given out without any corresponding expenditure; or, in other words, a perpetual source of mechanical work, commonly called a perpetual motion, would be possible. After farther consideration, however, the former conclusion appeared to be incontrovertible; but then, to avoid the absurdity of supposing that mechanical work could be got out of nothing, it occurred to me that it is necessary farther to conclude, that *the freezing point becomes lower as the pressure to which the water is subjected is increased.*

The following is the reasoning by which these conclusions are proved.

First, to prove that water at the freezing point may be converted into ice by a process solely mechanical, and yet without the final expenditure of any mechanical work:—Let there be supposed to be a cylinder, and a piston fitting water-tight to it, and capable of moving without friction. Let these be supposed to be formed of a substance which is a perfect non-conductor of heat; also, let the bottom of the cylinder be closed by a plate, supposed to be a perfect conductor, and to possess no capacity for heat. Now, to convert a given mass of water into ice without the expenditure of mechanical work, let this imaginary vessel be partly filled with air at 0° *C*,* and let the bottom of it be placed in contact with an indefinite mass of water, a lake for instance, at the same temperature. Now, let the piston be pushed towards the bottom of the cylinder by pressure from some external reservoir of mechanical work, which, for the sake of fixing our ideas, we may suppose to be the hand of an operator. During this process the air in the cylinder would tend to become heated on account of the compression, but it is constrained to remain at 0° by being in communication with the lake at that temperature. The change, then, which takes place is, that a certain amount of work is given from the hand to the air, and a certain amount of heat is given from the air to the water of the lake. In the next place, let the bottom of the cylinder be placed in contact with the mass of water at 0°, which is proposed to be converted into ice, and let the piston be allowed to move back to the position it had at the commencement of the first process. During this second process, the temperature of the air would tend to sink on account of the expansion, but it is constrained to remain constant at 0° by the air being in communication with the freezing water, which cannot change its temperature so long

* The centigrade thermometric scale is adopted throughout this paper.

as any of it remains unfrozen. Hence, so far as the air and the hand are concerned, this process has been exactly the converse of the former one. Thus the air has expanded through the same distance through which it was formerly compressed; and since it has been constantly at the same temperature during both processes, the law of the variation of its pressure with its volume must have been the same in both. From this it follows, that the hand has received back exactly the same amount of mechanical work in the second process as it gave out in the first. By an analogous reason it is easily shewn that the air also has received again exactly the same amount of heat as it gave out during its compression; and, hence, it is now left in a condition the same as that in which it was at the commencement of the first process. *The only change which has been produced then, is that a certain quantity of heat has been abstracted from a small mass of water at* 0°, *and dispersed through an indefinite mass at the same temperature, the small mass having thus been converted into ice.* This conclusion, it may be remarked, might be deduced at once by the application, to the freezing of water, of the general principle developed by Carnot, that no work is given out when heat passes from one body to another without a fall of temperature; or rather by the application of the converse of this, which of course equally holds good, namely that no work requires to be expended to make heat pass from one body to another at the same temperature.

Next, to prove that the freezing point of water is lowered by an increase of the pressure to which the water is subjected:—Let the imaginary cylinder and piston employed in the foregoing demonstration, be again supposed to contain some air at 0°. Let the bottom of the cylinder be placed in contact with the water of an indefinitely large lake at 0°; and let the air be subjected to compression by pressure applied by the hand to the piston. A certain amount of work is thus given from the hand to the air, and a certain amount of heat is given out from the air to the lake. Next, let the bottom of the cylinder be placed in communication with a small quantity of water at 0°, enclosed in a second imaginary cylinder similar in character to the first, and which we may call the water cylinder, the first being called the air cylinder; and let this water be, at the commencement, subject merely to the atmospheric pressure. Let, however, resistance be offered by the hand to any motion of the piston of the water cylinder which may take place. Things being in this state, let the piston of the air cylinder move back

to its original position. During this process, heat becomes latent in the air on account of the increase of volume, and therefore the air abstracts heat from the water, because the air and water, being in communication with one another, must remain each at the same temperature as the other, whether that temperature changes or not. The first effect of the abstraction of heat from the water must be the conversion of a part of the water into ice, an effect which must be accompanied with an increase of volume of the mass enclosed in the water cylinder. Hence, on account of the resistance offered by the hand to the motion of the piston of this cylinder, the internal pressure is increased, and work is received by the hand from the piston. Towards the end of this process, let the resistance offered by the hand gradually decrease, till, just at the end it becomes nothing, and the pressure within the water cylinder thus becomes again equal to that of the atmosphere. The temperature of the mass of partly frozen water must now be 0°, and the air in the other cylinder, being in communication with this, must have the same temperature. The air is therefore at its original temperature, and it has its original volume, or, in other words, it is in its original state. Farther, let the ice be converted, under atmospheric pressure, into water; the requisite heat being transferred to it from the lake by the mechanical process already pointed out, which involves no loss of mechanical work. Thus, now at the conclusion of the operation, the whole mass of water is left in its original state; and likewise, as has already been shewn, the air is left in its original state. Hence no work can have been developed by any change on the air and water, which have been used. But work has been given out by the piston of the water cylinder to the hand; and therefore an equal quantity of work must have been given from the hand to the air piston, as there is no other way in which the work developed could have been introduced into the apparatus. Now, the only way in which this can have taken place is by the air having been colder, while it was expanding in the second process, than it was while it was undergoing compression during the first. Hence it was colder than 0° during the course of the second process; or, in other words, *while the water was freezing, under a pressure greater than that of the atmosphere, its temperature was lower than* 0°.

The fact of the lowering of the freezing point being thus demonstrated, it becomes desirable, in the next place, to find what is the freezing point of water for any given pressure.

The most obvious way to determine this would be by direct experiment with freezing water. I have not, however, made any attempt to do so in this way. The variation to be appreciated is extremely small, so small in fact as to afford sufficient reason for its existence never having been observed by any experimenter. Even to detect its existence, much more to arrive at its exact amount by direct experiment, would require very delicate apparatus which would not be easily planned out or procured. Another, and a better mode of proceeding has, however, occurred to me: and by it we can deduce, from the known expansion of water in freezing, and the known quantity of heat which becomes latent in the melting of ice, together with data founded on the experiments of Regnault on steam at the freezing point, a formula which gives the freezing point in terms of the pressure; and which may be applied for any pressure, from nothing up to many atmospheres. The following is the investigation of this formula.

Let us suppose that we have a cylinder of the imaginary construction described at the commencement of this paper; and let us use it as an ice-engine analogous to the imaginary steam-engine conceived by Carnot, and employed in his investigations. For this purpose, let the entire space enclosed within the cylinder by the piston be filled at first with as much ice at 0° as would, if melted, form rather more than a cubic foot of water, and let the ice be subject merely to one atmosphere of pressure, no force being applied to the piston. Now, let the following four processes, forming one complete stroke of the ice-engine, be performed.

Process 1. Place the bottom of the cylinder in contact with an indefinite lake of water at 0°, and push down the piston. The effect of the motion of the piston is to convert ice at 0° into water at 0°, and to abstract from the lake at 0° the heat which becomes latent during this change. Continue the compression till one cubic foot of water is melted from ice.

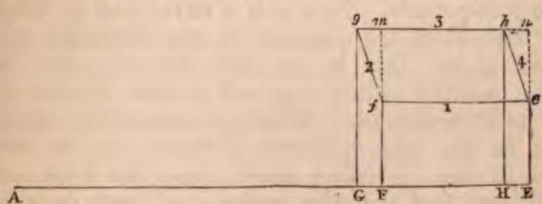
Process 2. Remove the cylinder from the lake, and place it with its bottom on a stand which is a perfect non-conductor of heat. Push the piston a very little farther down, till the pressure inside is increased by any desired quantity which may be denoted, in pounds on the square foot, by p. During this motion of the piston, since the cylinder contains ice and water, the temperature of the mixture must vary with the pressure, being at any instant the freezing point which cor-

responds to the pressure at that instant. Let the temperature at the end of this process de denoted by $-t^\circ C$.

Process 3. Place the bottom of the cylinder in contact with a second indefinitely large lake at $-t^\circ$, and move the piston upwards. During this motion the pressure must remain constant at p above that of the atmosphere, the water in the cylinder increasing its volume by freezing, since if it did not freeze, its pressure would diminish, and therefore its temperature would increase, which is impossible, since the whole mass of water and ice is constrained by the lake to remain at $-t^\circ$. Continue the motion till so much heat has been given out to the second lake at $-t^\circ$, as that if the whole mass contained in the cylinder were allowed to return to its original volume without any introduction or abstraction of heat, it would assume its original temperature and pressure. This, if Carnot's principles be admitted, as they are supposed be be throughout the present investigation, is the same as to say,—Continue the motion till all the heat has been given out to the second lake at $-t^\circ$, which was taken in during Process 2, from the first lake at 0°.

Process 4. Remove the cylinder from the lake at $-t^\circ$, and place its bottom again on the non-conducting stand. Move the piston back to the position it occupied at the commencement of Process 1. At the end of this fourth process the mass contained in the cylinder must, according to the condition by which the termination of Process 3 was fixed, have its original temperature and pressure, and therefore it must be in every respect in its original physical state.

By representing graphically in a diagram the various volumes and corresponding pressures, at all the stages of the four processes which have just been described, we shall arrive, in a simple and easy manner, at the quantity of work which is developed in one complete stroke by the heat which is transferred during that stroke from the lake at 0° to the lake at $-t^\circ$. For this purpose, let E be the position of the



piston at the beginning of Process 1; and let some distance,

such as EG, represent its stroke in feet, its area being made a square foot, so that the numbers expressing, in feet, distances along EG may also express, in cubic feet, the changes in the contents of the cylinder produced by the motion of the piston. Now, when 1·087 cubic feet of ice are melted, one cubic foot of water is formed. Hence, if EF be taken equal to ·087 feet, F will be the position of the piston when one cubic foot of water has been melted from ice, that is, the position at the end of Process 1, the bottom of the cylinder being at a point A distant from F by rather more than a foot. Let FG be the compression during Process 2, and HE the expansion during Process 4. Let *ef* be parallel to EF, and let E*e* represent one atmosphere of pressure; that is, let the units of length for the vertical ordinates be taken such that the number of them in E*e* may be equal to the number which expresses an atmosphere of pressure. Also let *gh* be parallel to EF, and let *fm* represent the increase of pressure produced during Process 2. Then the straight lines *ef* and *gh* will be the lines of pressure for Processes 1 and 3; and for the other two processes, the lines of pressure will be some curves which would extremely nearly coincide with the straight lines *fg* and *he*. For want of experimental data, the natures of these two curves cannot be precisely determined; but, for our present purpose, it is not necessary that they should be so, as we merely require to find the area of the figure *efgh*, which represents the work developed by the engine during one complete stroke, and this can readily be obtained with sufficient accuracy. For, even though we should adopt a very large value for *fm*, the change of pressure during Process 2, still the changes of volume *gm* and *hn* in Process 2 and Process 4 would be extremely small compared to the expansion during the freezing of the water; and from this it follows evidently that the area of the figure *efgh* is extremely nearly equal to that of the rectangle *efmn*, but *fe* is equal to *FE*, which is ·087 feet. Hence the work developed during an entire stroke is $\cdot 087 \times p$ foot-pounds. Now this is developed by the descent from 0° to $-t^\circ$ of the quantity of heat necessary to melt a cubic foot of ice; that is, by 4925 thermic units, the unit being the quantity of heat required to raise a pound of water from 0° to 1° centigrade. Next we can obtain another expression for the same quantity of work; for, by the tables deduced in the preceding paper from the experiments of Regnault, we find that the quantity of work developed by one of the same thermic units descending through one degree

about the freezing point, is 4·97 foot-pounds. Hence, the work due to 4925 thermic units descending from 0° to $-t^\circ$ is $4925 \times 4{\cdot}97 \times t$ foot-pounds. Putting this equal to the expression which was formerly obtained for the work due to the same quantity of heat falling through the same number of degrees, we obtain

$$4925 \times 4{\cdot}97 \times t = {\cdot}087 \times p.$$

Hence $$t = {\cdot}00000355\,p \,\ldots\ldots\ldots\ldots\ldots (1).$$

This, then, is the desired formula for giving the freezing point $-t^\circ$ centigrade, which corresponds to a pressure exceeding that of the atmosphere by a quantity p, estimated in pounds on a square foot.

To put this result in another form, let us suppose water to be subjected to one additional atmosphere, and let it be required to find the freezing point. Here p = one atmosphere = 2120 pounds on a square foot; and therefore, by (1),

$$t = {\cdot}00000355 \times 2120,$$

or $$t = {\cdot}0075.$$

That is, the freezing point of water, under the pressure of one additional atmosphere, is $-{\cdot}0075^\circ$ centigrade; and hence, if the pressure above one atmosphere be now denoted in atmospheres,* as units, by n, we obtain t, the lowering of the freezing point in degrees centigrade, by the following formula,

$$t = {\cdot}0075\,n \,\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

[The phenomena predicted by the author of the preceding paper, in anticipation of any direct observations on the freezing point of water, have been fully confirmed by experiment. See a short paper published in the *Proceedings of the Royal Society of Edinburgh* (Feb. 1850), and republished in the *Philosophical Magazine* for August 1850, under the title "The Effect of Pressure in Lowering the Freezing Point of Water experimentally demonstrated. By Prof. William Thomson."]

NOTE ON AN UNANSWERED PRIZE QUESTION.

By the Rev. THOMAS P. KIRKMAN, M.A.

THE Prize-question in the *Lady's and Gentleman's Diary* for 1844, was question 1733, by the editor:

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* The atmosphere is here taken as being the pressure of a column of mercury of 760 millimetres; that is 29·92, or very nearly 30 English inches.

"Determine the number of combinations that can be made out of n symbols, p symbols in each, with this limitation, that no combination of q symbols, which may appear in any one of them, may be repeated in any other."

All that has hitherto been offered towards a solution, so far as I know, is a paper of mine "On a Problem in Combinations," in the second volume (New Series) of this *Journal*, in which the special case of $p = 3$, $q = 2$, is completely discussed. The following three theorems are intended as a second contribution to the required answer, when $q = 2$.

THEOR. I. *If* r *be any prime number,* r^{m+1} *symbols can be formed into* $(\mathrm{r}^{m+1} - 1) : (\mathrm{r} - 1)$ *columns of* r*-plets, or combinations of* r *together, each column containing all the symbols, and so that every duad shall be once, and once only employed.*

THEOR. II. *If* r *be any prime number,* $(\mathrm{r}^2 + \mathrm{r} + 1)$ *symbols can be formed into* $(\mathrm{r}^2 + \mathrm{r} + 1)$ $(\mathrm{r} + 1)$*-plets, so that every duad shall be once, and once only employed.*

THEOR. III. *If* r *is any prime number, and if* $(\mathrm{r}^2 + \mathrm{r} + 1)$ *has no divisor less than* $(\mathrm{r} + 1)$, $(\mathrm{n} =)(\mathrm{r}^2 + \mathrm{r} + 1)(\mathrm{r} + 1)$ *symbols can be arranged into* $\mathrm{n}.(\mathrm{n} - 1) : \mathrm{r}(\mathrm{r} + 1)$ $(\mathrm{r} + 1)$*-plets, so that every duad shall be once, and once only employed.*

The proof of these propositions depends on that of the following:

THEOR. A. *If* $\mathrm{R} = \mathrm{A}^a\mathrm{B}^b\mathrm{C}^c\ldots\mathrm{L}^l\mathrm{M}^m\mathrm{N}$, *be any integer of which* A, B, C...L, M *are prime factors in decreasing order,* N *being any number not greater than* M, R *symbols can be thrown into* $(\mathrm{A}^a - 1) : (\mathrm{A} - 1)$ *columns of* A*-plets,* $+ \mathrm{A}^a.(\mathrm{B}^b - 1) : (\mathrm{B} - 1)$ *columns of* B*-plets,* $+ \mathrm{A}^a\mathrm{B}^b(\mathrm{C}^c - 1) : (\mathrm{C} - 1)$ *columns of* C*-plets,* $+\ldots+ \mathrm{A}^a\mathrm{B}^b\mathrm{C}^c\ldots\mathrm{L}^l\mathrm{M}^m$ *columns of* N*-plets, and this so that every column shall exhibit all the* R *symbols, and that every duad possible with these shall be once employed and repeated nowhere.*

To establish this, let us suppose that the theorem is true when R is any divisor towards the left of the above expression, as *e.g.* when $R = R' = A^aB^bC^r$, r being less than c. Write out C vertical rows thus, for a primary column:

$$\begin{array}{ccccc}
1_1 & 1_2 & 1_3\ldots\ldots & 1_{c-1} & 1_c \\
2_1 & 2_2 & 2_3\ldots\ldots & 2_{c-1} & 2_c \\
\vdots & \vdots & \ldots\ldots & \vdots & \vdots \\
(R'-1)_1 & (R'-1)_2 & \ldots\ldots & (R'-1)_{c-1} & (R'-1)_c, \\
R'_1 & R'_2 & \ldots\ldots & R'_{c-1} & R'_c.
\end{array}$$

Neglecting for a moment the subindices, write out the first of these vertical rows $R' - 1$ times, to form the first vertical rows of $R' - 1$ additional columns. Make the d^{th} of these rows into a column of R' C-plets, by completing each of the vertically written numbers in it, into an arithmetical series of C terms, having d for common difference; taking care meanwhile, in forming the series, to set down R' in place of every term kR', and e in place of every term $kR' + e$, $(e < R')$: there are thus completed $R' - 1$ columns of arithmetical (quasi) *cycles*, the cycles of the d^{th} column being formed with the common difference d, so that $R' - 1$ is the highest difference introduced.

The $C - 1$ added vertical rows in a column are all cyclical permutations of the first; for under the term $a + md$ stands always $a + 1 + md$; and if $a + md$ is set down R', $a + 1 + md$ will be set down 1. Attach now to every term of the e^{th} vertical row in every column the subindex e, and the $R' - 1$ additional columns are completed; and each evidently contains all the $R'C$ symbols.

I say now, that in these R' columns of R' C-plets each, no doublet is repeated; for let $m_e n_{e+i}$ be such a repeated pair (there has been no doublet employed of the form $m_e n_e$), occurring first in the column whose difference is b, and again in that whose difference is b_1. In the former of these columns, the numbers $m_e n_{e+i}$ stand for the pair

$$kR' + m = f + (e - 1)b,$$

$$k_1 R' + n = f + (e + i - 1)b,$$

and in the second, for the pair

$$lR' + m = f_1 + (e - 1)b_1,$$

$$l_1 R' + n = f_1 + (e + i - 1)b_1;$$

whence follows $(l_1 - l - k_1 + k)R' = i(b_1 - b)$.

The highest possible value of i is $C - 1$, and the highest possible of $(b_1 - b)$ is $R' - 1$, so that the right member of this equation is no multiple of R', which is absurd. Therefore no doublet is repeated in these R' columns, or of the form $m_e n_e$. By our supposition, we can form $(A^a - 1):(A - 1)$ columns of A-plets, with the symbols of our first vertical row, each column comprising them all; under these we can write as many columns of A-plets made with the symbols of our second vertical row; and so on, till we have completed $(A^a - 1):(A - 1)$ columns of A-plets, each containing

all the $R'C$ symbols, and shewing unrepeated duads only of the form $m_e n_e$.

By the same supposition, we can make with our first vertical row $A^a(B^b-1):(B-1)$ columns of B-plets, exhibiting duads of the form $m_e n_e$ only, which have not been employed in forming the A-plets, and thus we place under each other C sets of such columns, making $A^a.(B^b-1):(B-1)$ columns of B-plets, each column comprising all the $R'C$ symbols, and containing unrepeated duads only of the form $m_e n_e$. By the same supposition and process, we can make $A^aB^b.(C^r-1):(C-1)$ columns of C-plets, each column shewing the $R'C$ symbols, and unrepeated duads all of the form $m_e n_e$; adding now to these the $A^aB^bC^r$ columns of C-plets above formed with duads of the form $m_e n_i$, we have $A^aB^b(C^{r+1}-1):(C-1)$ columns of C-plets, each containing the $R'C$ symbols, and no repeated duad. Thus it is proved that the theorem is true for $R=A^aB^bC^{r+1}$, if it is so for $R=A^aB^bC^r$: and this would have been proved if C had been the k^{th} of the decreasing prime factors, or even if C^{r+1} had been the factor N. For the above absurdity stands for C not prime if $r=0$.

Now we know that the theorem is true for $R=A$; it is consequently true for $R=A^aB^0$ by the demonstration just given; and in like manner true for $R=A^aB^bC^0$, and so on; that is, the theorem is generally true, if only we have employed every duad once that can be made with R things.

The number of A-plets in a column is $R:A$; so that the duads exhausted in the $(A^a-1):(A-1)$ columns of A-plets are $\frac{1}{2}R.(A^a-1)$. The number exhausted in the $A^a.(B^b-1):(B-1)$ columns of B-plets is $\frac{1}{2}RA^a(B^b-1)$; the duads employed to make C-plets are $\frac{1}{2}R.A^a.B^b(C^c-1)$, &c.; and the total number of duads exhausted is

$$\tfrac{1}{2}R.\{(A^a-1)+A^a.(B^b-1)+A^a.B^b.(C^c-1)+\ldots$$
$$+A^a.B^b.C^c\ldots L^lM^m.(N-1)\}=\tfrac{1}{2}R.(R-1);$$

so that Theorem A is established.

Theorem I follows from theorem A by putting $N=1$, and $R=r^{m+1}$.

In theorem I let $m=1$; we can make $r+1$ columns each of r r-plets; and by prefixing a different new symbol in every column, we have $(r+1)$ columns of r $(r+1)$-plets each. The $r+1$ new symbols are combined with all the r^2 old ones, and make one $(r+1)$-plet more; hence we have (r^2+r+1) $(r+1)$-plets, made with r^2+r+1 symbols, and theorem II is demonstrated.

In theorem III (r^2+r+1) is prime; we can then, by theorem A, make with $(r^2+r+1)(r+1)$ symbols $(r^2+r+1)^2$ $(r+1)$-plets, and $(r+1)$ (r^2+r+1)-plets, without repeating any duad. Each of the latter multiplets breaks up by theorem II into (r^2+r+1) $(r+1)$-plets, so that we obtain $[(r^2+r+1)^2+(r^2+r+1)(r+1)=]\,n.(n-1):r(r+1)$ $(r+1)$-plets made with the $(r^2+r+1)(r+1)$ symbols, no duad being repeated; and as this number exhausts all the duads, theorem III is proved.

Theorem I, as well as the following which we are about to establish,

THEOR. B. 5.3^{m+1} *symbols can be arranged in* $\frac{1}{2}(5.3^{m+1}-1)$ *columns of triplets, each column containing all the symbols, and so that every duad shall be once and once only employed;*

is a partial solution of this,

PROBLEM D. *To assign the numbers* n, q, *and* r, *such that it shall be possible to arrange* nr *symbols in columns of* n r-*plets, each column containing all the symbols, so that no* q-*plet shall be more than once employed.*

It is to be remarked, that the solution of this problem will be auxiliar to that of the unanswered question, 1733 of the *Lady's and Gentleman's Diary.*

Let us suppose that theorem B is true for 5.3^m symbols. If we put $R=5.3^{m+1}$, $N=3$, in theorem A, we see that it is possible to form 5.3^m columns, each of 5.3^m triplets, so that each column shall contain the same 5.3^{m+1} symbols, no duad being twice employed. With the 5.3^m symbols which have all the subindex unity, we can, by hypothesis, make $\frac{1}{2}(5.3^m-1)$ columns of triplets, each column containing all the 5.3^m symbols of a primary vertical row; and this we can plainly do three times, placing the three sets each of $\frac{1}{2}(5.3^m-1)$ columns under each other, so as to make one set of $\frac{1}{2}(5.3^m-1)$ columns, each column containing all the 5.3^{m+1} symbols; then adding the 5.3^m columns formed before, we have $5.3^m+\frac{1}{2}(5.3^m-1)=\frac{1}{2}(5.3^{m+1}-1)$ columns, which were to be made, without repetition of any duad.

Now that theorem B is true for $m=0$, and therefore for all values of m, is plain, from inspection of the following 35 triplets, which are obtained by substituting in Q_{15}, at p. 195, the second for the first of the forms of D_8 given on the preceding page, of the second volume of this *Journal.* Q_{15} can be broken up, after this substitution, into the follow-

ing form; although this dispersion of the triplets, as they stand at page 195, is impracticable:

$$\begin{array}{lllllll}
a_1a_2a_3 & a_1b_1c_1 & a_1d_1e_1 & a_1b_2d_2 & a_1c_2e_2 & a_1b_3e_3 & a_1c_3d_3 \\
b_1b_2b_3 & a_2b_2c_2 & a_2d_2e_2 & a_2b_3d_3 & a_2c_3e_3 & a_2b_1e_1 & a_2c_1d_1 \\
c_1c_2c_3 & a_3d_3e_3 & a_3b_3c_3 & a_3c_1e_1 & a_3b_1d_1 & a_3c_2d_2 & a_3b_2e_2 \\
d_1d_2d_3 & b_3d_1e_2 & d_3b_1c_2 & b_1c_3e_2 & c_1b_3d_2 & b_2c_3d_1 & c_2b_3e_1 \\
e_1e_2e_3 & c_3d_2e_1 & e_3b_2c_1 & d_1c_2e_3 & e_1b_2d_3 & e_2c_1d_3 & d_2b_1e_3
\end{array}$$

This is the neatest mode of writing out the solution of a practical puzzle given by me at page 48 of the *Lady's and Gentleman's Diary*, 1850;

"Query 6. *Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two shall walk twice abreast;*"

which I should hardly deem worthy of mention in these pages, if it were not that it has excited, as I hear, some attention among a far higher class of readers than those for whom the first 48 pages of the *Diary* are intended. I hit upon this pretty puzzle four years ago, when preparing for this *Journal* the paper already mentioned; but I have in vain endeavoured to generalize the subject, and to shew, apart from all trial and comparison of triplets, *why* 5, 3, and 2 *must* be suitable values of n, r, and q, in the problem D above proposed. A solution of the puzzle, to be mathematical and not tentative, must inform us whether or no, and why, the values 7, 3, and 2 will also be suitable; and the same question of the values 11, 3, and 2 must be answered before trial and examination of triplets. All the solutions of this puzzle, that I have examined, can be reduced to the form above written, by changing the alphabetical order of the leading column, or the order of subindices in certain triplets of it, or by both. It is not easy to see the secret of the symmetry when 15 letters are employed, as in Mr. Cayley's solution in the *Phil. Mag.* for June 1850. It was owing to a passing remark of Mr. Salmon's, made in February last, to the effect that the use of subindices would probably bring out the true character of these arrangements, and facilitate their investigation, that I was first led to exhibit my solution in the form above given; and the result confirmed the truth of his valuable observation.

It is easy to point out, from this mode of writing the triplets, the truth of the charming extension of the puzzle which Mr. Sylvester has made (see Mr. Cayley's note, *Phil.*

Mag., June 1850), "To make the school walk out every day in the quarter, so that every three shall walk together."

The three final pairs of columns have three systems of subindices; permute cyclically these *systems*, and you get two more sets each of six columns. In each of the three sets of six columns permute cyclically the five letters, and you obtain the solution of Mr. Sylvester's puzzle.

Is it known that a similar feat can be achieved by nine ladies, to walk in threes till every three have walked together?

From the figure

$$\begin{array}{c} 1_1 1_2 1_3 \\ 2_1 2_2 2_3 \\ 3_1 3_2 3_3, \end{array}$$

write out the triplets

$$\begin{array}{cccc} 1_1 1_2 1_3 & 1_1 2_1 3_1 & 1_1 2_2 3_3 & 1_1 3_2 2_3 \\ 2_1 2_2 2_3 & 1_2 2_2 3_2 & 2_1 3_2 1_3 & 2_1 1_2 3_3 \\ 3_1 3_2 3_3 & 1_3 2_3 3_3 & 3_1 1_2 2_3 & 3_1 2_2 1_3\,; \end{array}$$

then, from the six figures

$$\begin{array}{cccccc} 1_1 1_2 2_1 & 2_1 2_2 3_1 & 3_1 3_2 1_1 & 3_1 1_1 1_2 & 1_1 2_1 2_2 & 2_1 3_1 3_2 \\ 2_2 2_3 3_1 & 3_2 3_3 1_1 & 1_2 1_3 2_1 & 1_3 2_1 2_2 & 2_3 3_1 3_2 & 3_3 1_1 1_2 \\ 3_2 3_3 1_3 & 1_2 1_3 2_3 & 2_2 2_3 3_3 & 2_3 3_2 3_3 & 3_3 1_2 1_3 & 1_3 2_2 2_3, \end{array}$$

arrangements may be made in exactly the same manner, so as to complete the solution. Can 25 be made to walk out in fives, till every five have walked together?

The *school-girl problem* may be shewn to depend on the combination of the triplets made with seven things, with the following curious arrangement of the duads made with eight things:

—	—	—	*hi*	*kl*	*mn*	*op*
—	*il*	*mo*	—	*np*	*hk*	—
—	*no*	*hl*	*mp*	—	—	*ik*
lp	—	*in*	*ko*	*hm*	—	—
im	—	*kp*	—	—	*lo*	*hn*
ho	*km*	—	*ln*	—	*ip*	—
kn	*hp*	—	—	*io*	—	*lm*.

It will be found difficult to imitate this arrangement with more than eight things.

I must not omit to add, that I had no solution of the problem D, except the cases of $nr = 9$ and $nr = 15$, until I saw the following arrangement of 27 things by the Rev. James Mease, A.M., of Freshford, Kilkenny, who sent it

to me in the month of March (1850), accompanied by the enunciation of the theorem following:

"3^{n+1} symbols can be formed into $\frac{1}{2}(3^{n+1}-1)$ columns of triplets, each column containing all the symbols, so that every doublet shall be once, and once only, employed." Rev. J. Mease.

It will be seen by the reader how much I am indebted to the valuable hint supplied by Mr. Mease's discovery.

abc	*aei*	*ahp*	*alv*	*ao*β	*arf*	*aum*	*axs*	*aay*
def	*dhm*	*dls*	*doy*	*drc*	*dui*	*dxp*	*dav*	*db*β
ghi	*glp*	*gov*	*gr*β	*guf*	*gxm*	*gas*	*gby*	*gec*
klm	*kos*	*kry*	*kuc*	*kxi*	*kap*	*kbv*	*ke*β	*khf*
nop	*nrv*	*nu*β	*nxf*	*nam*	*nbs*	*ney*	*nhc*	*nli*
qrs	*quy*	*qxc*	*qai*	*qbp*	*qev*	*qh*β	*qlf*	*qom*
tuv	*tx*β	*taf*	*tbm*	*tes*	*thy*	*tlc*	*toi*	*trp*
wxy	*wac*	*wbi*	*wep*	*whv*	*wl*β	*wof*	*wrm*	*wus*
*za*β	*zbf*	*zem*	*zhs*	*zly*	*zoc*	*zri*	*zup*	*zxv*

adg	*anz*	*awq*	*akt*
knq	*kwg*	*kdz*	*dnw*
twz	*tdq*	*tng*	*gqz*
beh	*boa*	*bxr*	*blu*
lor	*lxh*	*lea*	*eox*
uxa	*uer*	*uoh*	*hra*
cfi	*cp*β	*cys*	*cmv*
mps	*myi*	*mf*β	*fpy*
*vy*β	*vfs*	*vpi*	*is*β.

The first nine columns are formed by obvious cyclical permutations in the second and third vertical rows.

Croft Rectory, near Warrington, Lancashire,
August 23, 1850.

ON THE INTERSECTIONS, CONTACTS, AND OTHER CORRELATIONS OF TWO CONICS EXPRESSED BY INDETERMINATE COORDINATES.

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Let $U=0$, $V=0$ be two homogeneous equations of the second degree, and with real coefficients between the same three variables ξ, η, ζ.

The direct and most general mode of determining the intersections of the conics expressed by these equations would be to make

$$a\xi + b\eta + c\zeta = t,$$
$$a'\xi + b'\eta + c'\zeta = u:$$

eliminating ξ, η, ζ between the four equations in which they appear, there results a biquadratic equation between t and u. The nature of the intersections will depend upon the nature of the roots of this biquadratic; and thus the conditions may be expressed analytically, which will represent the several cases of all the intersections being real or all imaginary, or one pair real and the other imaginary. These analytical conditions will depend upon the signs of certain functions of the coefficients of the given and the *assumed* equations being of an assigned character; my endeavour has been to obtain conditions of a character perfectly symmetrical and free from the coefficients arbitrarily introduced.

In this research I have only partially succeeded, but the method employed, and some of the collateral results, will, I think, be found of sufficient interest to justify their appearance in the pages of this *Journal*.

Adopting Mr. Cayley's excellent designation, let the four points of intersection of the two conics be called a quadrangle. This quadrangle will have three pairs of sides; the intersections of each pair, from principles of analogy, I call the vertices of the quadrangle. Then, inasmuch as the four sets of ratios $\xi : \eta : \zeta$, corresponding with the four sets of the ratio $t : u$, must be so related that we may always make

$$\frac{\xi_1}{\zeta_1} = a + b\sqrt{(-1)}, \quad \frac{\eta_1}{\zeta_1} = c + d\sqrt{(-1)},$$
$$\frac{\xi_2}{\zeta_2} = a - b\sqrt{(-1)}, \quad \frac{\eta_2}{\zeta_2} = c - d\sqrt{(-1)},$$
$$\frac{\xi_3}{\zeta_3} = \alpha + \beta\sqrt{(-1)}, \quad \frac{\eta_3}{\zeta_3} = \gamma + \delta\sqrt{(-1)},$$
$$\frac{\xi_4}{\zeta_4} = \alpha - \beta\sqrt{(-1)}, \quad \frac{\eta_4}{\zeta_4} = \gamma - \delta\sqrt{(-1)},$$

we may easily draw the following conclusions.

If all the four points of the quadrangle of intersection are real, the three vertices and the three pairs of sides are all real. If only two points of the quadrangle are real, one vertex and one of the three pairs of sides will be real; the other two vertices and two pairs of sides being imaginary.

If all four points of the quadrangle are unreal, one pair of sides will be real and the other two pairs imaginary, as in the last case; but all the three vertices will remain real, as in the first case. Hence we have a direct and simple criterion for distinguishing the case of *mixed* intersection from intersection wholly real or wholly imaginary; namely, that the cubic equation of the roots of which the coordinates of the vertices are real linear functions shall have *a pair of imaginary roots*. This is the sole and unequivocal condition required.

The equation in question is, or ought to be, well known to be the determinant in respect to ξ, η, ζ of $\lambda U + \mu V$. In fact, if we write

$$U = a\xi^2 + b\eta^2 + c\zeta^2 + 2a'\eta\zeta + 2b'\zeta\xi + 2c'\xi\eta,$$
$$V = \alpha\xi^2 + \beta\eta^2 + \gamma\zeta^2 + 2\alpha'\eta\zeta + 2\beta'\zeta\xi + 2\gamma'\xi\eta,$$
$$\lambda U + \mu V = (a\lambda + \alpha\mu)\,\xi^2 + \text{etc.}$$
$$= A\xi^2 + B\eta^2 + C\zeta^2 + 2A'\eta\zeta + 2B'\xi\zeta + 2C'\xi\eta,$$

the ratios of the coordinates ξ, η, ζ of the vertex of $\lambda U + \mu V$ may easily be shewn to be identical with

$$AB - C'^2 : C'A' - B'B : B'C' - A'A,$$

and will be real or imaginary as $\lambda : \mu$ is one or the other.

If then the cubic equation in $\lambda : \mu$, viz. $\underset{\xi\eta\zeta}{\square}(\lambda U + \mu V) = 0$, has a pair of imaginary roots; *i.e.* if $\underset{\lambda\mu}{\square}\ \underset{\xi\eta\zeta}{\square}(\lambda U + \mu V)$ is a positive quantity, the intersections of U and V are of a mixed kind, *i.e.* the two conics have two real points in common.

I may remark here, *en passant*, that if we form the biquadratic equation in t and u, $\phi(t, u) = 0$ from the equations

$$U = 0,$$
$$V = 0,$$
$$a\xi + b\eta + c\zeta = t,$$
$$a'\xi + b'\eta + c'\zeta = u,$$

and if any reducing cubic of this equation be $P(\theta, \omega) = 0$ the determinant of $P(\theta, \omega)$ must, from what has been shewn above, be identical with $\underset{\lambda\mu}{\square}\ \underset{\xi\eta\zeta}{\square}(\lambda U + \mu V)$ multiplied by some *squared function* of the extraneous coefficients

$$a, b, c;\ a', b', c'.$$

If $\square\,\square(\lambda U + \mu V)$ is a negative quantity, it remains to distinguish between the cases of the conics intersecting really in four points or not at all.

The most obvious mode of proceeding to distinguish between purely real and purely imaginary intersections would be as follows. Let $\lambda_1, \mu_1; \lambda_2, \mu_2; \lambda_3, \mu_3$, be the three sets of values of λ, μ which satisfy the equation

$$\square(\lambda U + \mu V) = 0$$

and make

$$\begin{array}{lll} A_1 = a\lambda_1 + \alpha\mu_1, & A_2 = a\lambda_2 + \alpha\mu_2, & A_3 = a\lambda_3 + \alpha\mu_3, \\ C_1 = c\lambda_1 + \gamma\mu_1, & C_2 = c\lambda_2 + \gamma\mu_2, & C_3 = c\lambda_3 + \gamma\mu_3, \\ B_1' = b'\lambda_1 + \beta'\mu_1, & B_1^2 = b'\lambda_2 + \beta'\lambda_2, & B_1^3 = b'\lambda_3 + \beta'\mu_3, \\ A_1C_1 - B_1^2 = e_1, & A_2C_2 - B_2^2 = e_2, & A_3C_3 - B_3^2 = e_3. \end{array}$$

Now if the equation

$$A\xi^2 + B\eta^2 + C\zeta^2 + 2A'\eta\zeta + 2B'\xi\xi + 2C'\xi\eta = 0$$

represent a pair of conics, it may be thrown into the form

$$Au^2 + \frac{AC - B'^2}{A}v^2 = 0,$$

where u and v are linear functions of ξ, η, ζ, and the conics will be real or imaginary, according as $B'^2 - AC$ is positive or negative; hence one or else all of the quantities e_1, e_2, e_3, will be necessarily negative, and the intersections will be all real or all imaginary, according as all three are negative or only one is so. A cubic equation in e may be formed containing e_1, e_2, e_3 as its roots by eliminating between the equations

$$e = AC - B_1^2; \quad \square(\lambda U + \mu V) = 0,$$

and the conditions for the reality of the intersections will be that all four coefficients of this cubic shall be of the same sign, which in reality amount only to two, since the first and last must in all cases have the same sign.

The same objection however of want of symmetry and consequent irrelevancy and complexity attaches to this as much as to the method originally proposed. The following treatment of the question relieves the objection of want of symmetry as far as the coefficients of the same equation are concerned, but in its practical application necessitates an arbitrary and therefore unsymmetrical election to be made between the two sets of coefficients appertaining to the two equations. It is however, I think, too curious and sug-stive to be suppressed.

I observe that if the four intersections are all real, an aginary conic cannot be drawn through them; for the

equation to an imaginary conic may always be reduced to the form $Ax^2 + By^2 + Cz^2 = 0$, where A, B, C are all positive and can therefore have at utmost one real point. Consequently the case of total non-intersection is distinguishable from that of complete intersection by the peculiarity that in the one case μ may be so taken that $U + \mu V = 0$ shall represent an imaginary conic, *i.e.* $U + \mu V$ will be a function whose sign never changes for real values of ξ, η, ζ, whereas in the latter case no value of μ will make $U + \mu V = 0$, the equation to an imaginary conic, and therefore $U + \mu V$ will have values on both sides of zero. On the other hand, it is obvious that an infinite number of real as well as unreal conics may be drawn through four imaginary points of intersection. Consequently if we make $U + \mu V = 0$ (supposing the intersections of U and V to be imaginary), there will be a range or ranges of values of μ consistent, and another range or ranges of values of μ inconsistent with real values of ξ, η, ζ; in other words, $U \pm \mu V = 0$ treated as an equation between the four variables ξ, η, ζ, μ, will give one or more maxima or minima values of μ in the case supposed, but no such values when the intersections are two or all of them real.

To determine these values of μ, let $d\mu = 0$; then we have

$$\frac{d}{d\xi}(U - \mu V) = 0,$$

$$\frac{d}{d\eta}(U - \mu V) = 0,$$

$$\frac{d}{d\zeta}(U - \mu V) = 0,$$

i.e. $$\underset{\xi\eta\zeta}{\square}(U - \mu V) = 0.$$

In order that any value of μ found from this equation may be a maximum or minimum, Lagrange's condition requires that

$$\left(h\frac{d}{d\xi} + k\,\frac{d}{d\eta} + l\,\frac{d}{d\zeta}\right)^2 \mu$$

may be a function of unchangeable sign.

Now $$\frac{dU}{d\xi} = \mu\,\frac{dV}{d\xi} + V\,\frac{d\mu}{d\xi},$$

therefore since $d\mu = 0$,

$$\frac{d^2U}{d\xi^2} = \mu\,\frac{d^2V}{d\xi^2} + V\,\frac{d^2\mu}{d\xi^2}.$$

13. Hence $$\frac{d^2\mu}{d\xi^2} = \frac{1}{V}\left(\frac{d}{d\xi}\right)^2\{U-\mu V\};$$

similarly $$\frac{d}{d\xi}\cdot\frac{d}{d\eta} = \frac{1}{V}\frac{d}{d\xi}\cdot\frac{d}{d\eta}\{U-\mu V\},$$

&c. &c. &c.

Making now as before

$$U = a\xi^2 + b\eta^2 + \&c.,$$

$$V = \alpha\xi^2 + \beta\eta^2 + \&c.,$$

$$a - \mu\alpha = A \quad b - \mu\beta = B, \ \&c.,$$

the condition for μ a root of $\square\{U-\mu V\} = 0$, giving μ a maximum or minimum, may be expressed by saying that

$$Ah^2 + Bk^2 + Cl^2 + 2A'kl + 2B'hl + 2C'hk$$

shall be unchangeable in sign for all real values of h, k, l.

The above quantity, by virtue of the equation $\square = 0$, is always the product of two linear functions. Hence we see, as above indicated, that if all these pairs are real, *i.e.* if all the points of intersection of U and V are real, there is no maximum or minimum value of μ; but if only one pair be real and the other two pairs be imaginary, *i.e.* if all the four intersections are imaginary, then two of the values of μ, namely those corresponding to the imaginary pairs, are real maxima or minima values of μ, but the third is illusory.

Now I shall shew that if $V = 0$ is a *real* conic, but the intersections of U and V are all unreal, the value of μ which makes $U + \mu V$ the product of real linear functions of ξ, η, ζ, is always one or the other *extreme* of the three values of μ which satisfy the equation

$$\square(U - \mu V) = 0.$$

Assume as the three axes of coordinates the three lines joining the vertices of the quadrangle each with each, the two non-intersecting conics may evidently be written under the form $$U = c(x^2 + y^2) - e(y^2 + z^2) = 0,$$

$$V = -\gamma(x^2 + y^2) + \epsilon(y^2 + z^2) = 0;$$

these equations being only other modes of writing

$$U = Ax^2 + By^2 + Cz^2,$$

$$V = A'x^2 + B'y^2 + C'z^2,$$

in which A, B, C; A', B', C' will be real, because by hypothesis $\square(U + \mu V) = 0$ has all its roots real.

Hence x, y, z are linear functions of ξ, η, ζ, and consequently, by a simple inference from a theorem of Prof. Boole,* the roots of $\underset{\xi\eta\zeta}{\square}\{U + \mu V\}$ are identical with those of

$$\underset{xyz}{\square}\{U + \mu V\} = 0.$$

These latter are evidently $\frac{c}{\gamma}, \frac{e}{\varepsilon}, \frac{c-e}{\gamma-\varepsilon}$; the third of which is the one which makes $U + \mu V$ the product of two *real* linears, for we have

$$\gamma U + cV = (c\varepsilon - \gamma e)(y^2 + z^2),$$
$$\varepsilon U + eV = (\varepsilon c - e\gamma)(x^2 + y^2),$$
$$(\gamma - \varepsilon)U + (c - e)V = (c\varepsilon - e\gamma)(z^2 - x^2).\dagger$$

Now
$$\frac{c}{\gamma} - \frac{c-e}{\gamma-\varepsilon} = \frac{e\gamma - c\varepsilon}{\gamma(\gamma-\varepsilon)},$$
$$\frac{e}{\varepsilon} - \frac{c-e}{\gamma-\varepsilon} = \frac{e\gamma - c\varepsilon}{\varepsilon(\gamma-\varepsilon)};$$

and ε, γ are supposed to have the same sign, as otherwise V would be an unreal conic; hence the ascending or descending order of magnitudes of the three values of λ follows the scale $\frac{c}{\gamma}, \frac{e}{\varepsilon}, \frac{c-e}{\gamma-\varepsilon}$, as was to be shewn.

Imagine now lengths reckoned on a line corresponding to all values of μ from $-\infty$ to $+\infty$, and mark off upon this line by the letters A, B, C, the lengths corresponding with the three roots of $\square(U + \mu V) = 0$. Then observing that when $\mu = \pm\infty$, $U + \mu V$ is of the same nature as V, and is therefore a possible conic by hypothesis, and agreeing to understand by a possible and impossible region of μ, a range of values for which $U + \mu V$ corresponds to a possible and impossible conic respectively, one or the other of the annexed schemes will represent the circumstances of the case supposed:

$-\infty$ Poss. Reg. A Imposs. Reg. B Poss. Reg. C Poss. Reg. $+\infty$

$-\infty$ Poss. Reg. A Poss. Reg. B Imposs. Reg. C Poss. Reg. $+\infty$

But in either scheme it is essential to observe that the *middle* root of $\square(U + \mu V) = 0$ divides a possible from an impossible region; and therefore if we can find n, ν, any two values lying between the first and second and second and third roots of

* See Postcript. † $z^2 - x^2 = 0$ of course represents a *real* pair of lines.

the above equation arranged in order of their magnitude, one of the two equations $U + \nu V = 0$, $U + nV = 0$, will represent a possible and the other an impossible conic: one such couple of values may always be found by taking the roots of the quadratic equation

$$\frac{d}{d\mu} \square \{U + \mu V\} = 0.$$

Hence calling the two roots thereof m and M, we see (which is in itself a theorem) that one at least of the conics $U + mV = 0$, $U + MV = 0$, must be a possible conic, provided only that $V = 0$ be a possible conic: if both $U + mV$ and $U + MV$ are possible conics, the intersections of U and V are all real, and if not, not.* The criteria for distinguishing possible from impossible conics being well known need not be stated in this place.

We may of course proceed analogously by forming the two conics $lU + V$, $LU + V$, where l and L are the roots of $\frac{d}{d\lambda} \square \{\lambda U + V\} = 0$ upon the supposition of $U = 0$ being a possible conic.

If either of the two U and V be not possible, their intersections are of course impossible, and the question is already decided.

It will be seen as pre-indicated that this method only fails in symmetry because of the choice between the couples m, M, and l, L. But moreover a perfect method for the discrimination of the two cases of *unmixed* intersection one from the other should (perhaps?) require the application of only a single test (in lieu of the two conditions which the above method supposes), over and above the condition which expresses the fact of the intersections being so unmixed. Such more perfect method I have not yet been able to achieve.

Another interesting question of intersections remains to be discussed, viz. supposing the two conics are known to be

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* It must be well observed however that the possibility of the conics $U + mV$ and of $U + MV$ does not imply the reality of the intersections unless the conic V is known to be possible.

For if V be impossible ε and γ have opposite signs, and therefore $\frac{c - e}{\gamma - \varepsilon}$ is intermediate between $\frac{c}{e}$ and $\frac{\gamma}{\varepsilon}$, and the scheme for μ will be as here annexed:

$-\infty$ ☜ Impossible. A Possible. B Possible. C Impossible. ☞ $+\infty$

so that $U + mV$ and $U + MV$ will both represent possible conics.

non-intersecting, how are we to ascertain if they are external to one another, or if one contains the other? In order to settle this point we must first establish a criterion for determining whether a given *point* is internal or external to a given conic; the point being in general said to be external when two real tangents can be drawn from it to the curve, and internal when this cannot be done.

Let now

$$\phi(x, y, z) = ax^2 + by^2 + cz^2 + 2a'yz + 2b'zx + 2c'xy = 0,$$

be the equation to any conic: l, m, n the coordinates of any point. Let

$$A = bc - a'^2, \quad B = ca - b'^2, \quad C = ab - c'^2,$$

$$A' = ca' - b'c', \quad B' = bb' - c'a', \quad C' = cc' - a'b'.$$

Then the reciprocal equation to the conic is

$$A\xi^2 + B\eta^2 + C\zeta^2 + 2A'\eta\zeta + 2B'\zeta\xi + 2C'\xi\eta = 0,$$

and in making $l\xi + m\eta + n\zeta = 0$, the ratios of ξ, η, ζ must be real if the tangents drawn from l, m, n are real: this will be found to imply that the determinant

$$\left.\begin{matrix} A, & C', & B', & l \\ C', & B, & A', & m \\ B', & A', & C, & n \\ l, & m, & n, & o \end{matrix}\right\} \text{ shall be negative.*}$$

This determinant may be shewn† to be equal to the product of the determinant

$$\left.\begin{matrix} a, & c', & b' \\ c', & b, & a' \\ b', & a, & c' \end{matrix}\right\} \text{ by the quantity}$$

$$al^2 + bm^2 + cn^2 - 2a'mn - 2b'ln - 2c'lm,$$

i.e. is equal to $\phi(l, m, n) \times \square$.

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* See theorem of the "Diminished Determinant" in Postscript to this paper.

† As we know *à priori* by virtue of a theorem given by M. Cauchy, and which is included as a particular case in a theorem of my own, relating to Compound Determinants, *i.e.* Determinants of Determinants, which will take its place as an immediate consequence of my fundamental Theorem given in a Memoir about to appear. The well-known rule for the multiplication of Determinants, is also a direct and simple consequence from my theorem on Compound Determinants, which indeed comprises, I believe, in one glance, all the heretofore existing Doctrine of Determinants.

Hence l, m, n is internal or external to $\phi(x, y, z)$ according as $\phi(l, m, n)$ and $\square\phi$ have the same or contrary sign.

If $\phi(l, m, n) = 0$, the point lies on the conic, and the point is neither internal nor external; if $\square\phi = 0$, the conic becomes a pair of straight lines, and no point can be said either to be within or without such a system. Hence our criterion fails, as it *ought to do*, just in the very two cases where the distinction vanishes. I believe that this criterion is here given for the first time.

To return to the two non-intersecting conics. Let us again throw them under the form

$$U = (x^2 + y^2) - e^2(z^2 + y^2),$$
$$V = k(x^2 + y^2) - k\epsilon^2(z^2 + y^2),$$

e and ϵ being real, *i.e.* U and V being both functions corresponding to possible conics. Suppose U external to V; then *any point* in U is an external point to V.

Take in U either of the two points represented by the equations $y = 0$, $x^2 = e^2z^2$; substituting these values of y and x, V becomes $K(e^2 - \epsilon^2)z^2$, and $\square V$ becomes $-k^3\epsilon^2(1-\epsilon^2)$; therefore $(1-\epsilon^2)(e^2-\epsilon^2)$ must be positive, *i.e.* ϵ^2 must be one of the extremes of the three values $1, e^2, \epsilon^2$. In like manner, if V is external to U, e will be also one of the extremes of the same three quantities; and hence, if the two conics are mutually external, unity will be the middle magnitude of the group $e^2, 1, \epsilon^2$.

Now the three roots of $\square(V + \lambda U) = 0$, are

$$\lambda = -k, \quad \lambda = -k\frac{\epsilon^2}{e^2}, \quad \lambda = -k\frac{1-\epsilon^2}{1-e^2}.$$

Hence if U and V be without one another, or, as it may be termed, are extra-spatial, the third value of λ will be of a different sign from the first two; but if the two conics be co-spatial, *i.e.* if one includes the other, all the three values of λ will have the same sign. Hence we have the following elegant criterion of co-spatiality of two possible conics expressed by the equations $U = 0$, $V = 0$, between indeterminate coordinates ξ, η, ζ; the coefficients of the cubic function $\underset{\xi\eta\zeta}{\square}(\lambda U + \mu V)$ must give only changes or only continuations of sign.

If this test be not satisfied, it will remain to determine which of the two conics contains, and which is contained by the other. Let U contain V, then the order of magnitudes

will be $1, e^2, \epsilon^2$; therefore $k\,\frac{1-\epsilon^2}{1-e^2}$ is greater than k, and therefore $k\,\frac{1-\epsilon^2}{1-e^2}$, which is that root of the equation $\Box(V+\lambda U)=0$, which is always one or the other of the extremes, is the *greatest* of the three. Hence the scheme for the impossible and possible regions of λ will be as below:

$-\infty$ Poss. A Impos. B Poss. C Poss. $+\infty$

Hence if the two roots of $\frac{d}{d\lambda}\{V+\lambda U\}=0$ be l and L, and of the two conics $V+lU=0$, $V+LU=0$, the former be the possible, and the latter the impossible one, U contains V, or is contained in it according as l is greater or less than L.

Observe that if U and V be non-cospatial, so that the three values of μ in $\Box(U+\mu V)=0$ have not all the same sign and consequently zero lies between the greatest and least of them, it will not be necessary to make trial of the characters of the two curves $U+mV=0$, and $U+MV=0$, in order to ascertain whether U and V intersect or not; for it will be sufficient to find which of the two quantities m and M substituted for μ in $\Box(U+\mu V)$ causes it to have the opposite sign to $\Box(U+0V)$, *i.e.* $\Box U$, and this one of the two it is, if either, which will make $U+\mu V$ an impossible conic, and will thus alone serve to determine whether the intersections of U and V are unreal, or the contrary.

It might be a curious question to consider whether, in a certain sense, conics not both possible may not be said to lie one within or without the other. Upon general logical grounds, I think it not improbable that two impossible conics might be discovered *each to contain the other*; but this is an inquiry which I have not had leisure to enter upon.

I have thus far supposed the roots of $\Box(\lambda U+V)=0$ to be all distinct from one another. I now approach the discussion of the contact of two conics, in which event two or more of the roots will be equal. The condition for simple contact is evidently $\underset{\lambda\mu}{\Box}, \underset{\xi\eta\zeta}{\Box}(\lambda U+\mu V)=0$.

The unpaired value of λ in $\Box(\lambda U+V)$ makes $\lambda U+V$ an impossible pair of lines, and therefore, in the scheme for λ drawn as above, will separate the possible from the impossible region.

Whether the conics intersect in two real or two unreal points, besides the point of contact, will be known at once

by ascertaining whether $U + \mu V = 0$ represents two real or two imaginary lines. If the latter, the two curves lie dos-à-dos or one within the other, according as the successions of sign in $\square(\lambda U + V)$ are all of the same kind or not; if they be all of the same kind, one will include the other, viz. U will include V if the equal roots are greater, and be included in it if they be less than the unequal one. This last conclusion however, it should be observed, is inferred upon the principle of continuity, by making two values of λ approach indefinitely near to one another, but cannot be strictly deduced from the equations given for U and V applicable to the general case, in which the axes of coordinates are the three axes joining the vertices; since these latter, in the case supposed, reduce to two only, and consequently such representation of U and V becomes illusory.

If all three values of λ are equal, the three vertices come together, and hence the two conics will have three consecutive points in common, *i.e.* will have the same circle of curvature. On this supposition the two curves act at the point of contact, and all four points of intersection are of course real.

The classification of contacts between two conics may be stated as follows:

Simple contact = one case.

Second degree contact = two cases, viz. common curvature or double contact.

Third degree contact = one case, viz. contact in four consecutive points.

These four cases of course correspond to the several suppositions of there being two equal roots, three equal roots, two pairs of equal roots, or four equal roots in the biquadratic equation obtained between two variables by elimination performed in any manner between the given equations in the two conics.

The first species and the first case of the second species have been already disposed of. I proceed to assign the conditions appertaining to the second case of the second species, when U and V have a double contact.

Let A, A', B, B' be the two pairs of coincident points in which the conics are supposed to meet; either pair of lines $AB, A'B'$, and $AB', A'B$, becomes a coincident pair. Hence such a value of μ can be found as will make $U + \mu V$ the *square of a linear function of* ξ, η, ζ. If therefore we make

$U + \mu V = W$, and form the determinant

$$\begin{vmatrix} \frac{d^2W}{d\xi^2}, & \frac{d^2W}{d\xi d\eta}, & \frac{d^2W}{d\xi d\zeta}, & p \\ \frac{d^2W}{d\eta d\xi}, & \frac{d^2W}{d\eta^2}, & \frac{d^2W}{d\eta d\zeta}, & q \\ \frac{d^2W}{d\zeta d\xi}, & \frac{d^2W}{d\zeta d\eta}, & \frac{d^2W}{d\zeta^2}, & r \\ p, & q, & r, & 0 \end{vmatrix}$$

$$= Ap^2 + Bq^2 + Cr^2 + 2Fqr + 2Grp + 2Hpq,$$

where all the coefficients are quadratic functions of λ, and make $A = 0,\ B = 0,\ C = 0,\ F = 0,\ G = 0,\ H = 0;$

each of these six equations in λ will have one and the same root in common.

It is, however, enough to select any three; if these vanish together for any value of λ, the remaining three must also vanish. This is a simple application of a general law* which will appear in a forthcoming memoir on Determinants and Quadratic Forms, of which this paper is to be considered as an accidental episode.

Take now any three of the six equations which for the sake of generality call $P = 0$, $Q = 0$, $R = 0$. The hypothesis of double contact requires that P and Q, Q and R, R and P shall all have a factor in common; but these conditions are not sufficiently explicit for our present object, since P, Q, R might be of the form

$$\kappa(\lambda - a)(\lambda - b), \quad \kappa'(\lambda - b)(\lambda - c), \quad \kappa''(\lambda - c)(\lambda - a),$$

and would thus satisfy the conditions above stated, without P, R, P having a common factor. A sufficient criterion is that $fQ + gR$ and P shall have a common factor for all values of f and g.

Let then the resultant of $fQ + gR$ and P be

$$Lf^2 + Mfg + Ng^2,$$

we must have $\qquad L = 0,\ M = 0,\ N = 0,$

where $\qquad L$ is the resultant of P and Q,

$\qquad N$ . . . . R and Q;

and M is a new function, which if we call $Q = \phi(\lambda)$, $R = \psi(\lambda)$,

* For statement of this law called the Homaloidal Law, see *Philosophical Magazine* of this Month "On Certain Additions, &c."

and suppose (a) and (b) to be the two roots of $P = 0$, is easily seen to be equal to $\phi a.\psi b + \phi b.\psi a$. This I call the connective of $P.Q$ and $P.R$.

L, M, N may conveniently be denoted by the forms

$$P.Q, \; P.R, \; Q.P.R.$$

We may now take more generally

$$aP + bQ + cR,$$

$$\alpha P + \beta Q + \gamma R,$$

which will have a factor in common for all values of $a, b, c, \alpha, \beta, \gamma$.

I am indebted to Mr. Cayley for the remark that the resultant of these two functions is a new quadratic function, which, according to my notation just given, may be put under the form—

$$PQ(a\beta - b\alpha)^2 + QR(b\gamma - c\beta)^2 + RP(c\alpha - a\gamma)^2$$
$$+ PRQ(b\gamma - c\beta)(c\alpha - \alpha\gamma) + QPR(c\alpha - a\gamma)(a\beta - b\alpha)$$
$$+ RQP(a\beta - b\alpha)(b\gamma - c\beta).$$

Ternary systems of the six coefficients formed upon the type of (PQ, PQR, QR), I call *complete* systems, because the three functions included in such a system equated severally to zero, imply that the remaining three coefficients are all zero. Such a system as (PQ, QR, RP) I term an *incomplete* ternary system as not drawing with it the like implication. Probably (?) we should find on investigation that PRQ, QPR, RQP, would also be an incomplete system, but that systems formed after the type of PRQ, RQ, RQP are complete. This however is only matter of conjecture, as I have been too much occupied with other things to enter upon the inquiry. The distinct types of ternary systems are altogether six in number, viz. four of a symmetrical species,

$$\begin{array}{ccc} PQ, & QR, & RP, \\ PRQ, & QPR, & RQP, \\ PQ, & PQR, & QR, \\ PRQ, & RQ, & RQP; \end{array}$$

and two of an insymmetrical species, viz.

$$\begin{array}{ccc} PQ, & PQR, & PR, \\ PRQ, & RQ, & QPR.^* \end{array}$$

* PQ, QR, RP, may be compared in a general way with the angles, and PRQ, QPR, RQP, with the sides of a triangle.

If instead of confining ourselves to three out of the six original quantities, A, B, C; F, G, H, we take them all into account, and write down the resultant of

$$aA + bB + cC + fF + gG + hH,$$
$$\alpha A + \beta B + \gamma C + \phi F + \chi G + \eta H;$$

we shall obtain a quadratic function of 15 variables (not however all independent) having 120 coefficients, all of which must be zero. It would be extremely interesting to determine how many *complete* ternary groups can be formed out of these 120 terms.

It will be recollected that we have assigned as the condition of contact in three consecutive points, that a certain cubic equation shall have all its roots real. Now, as well remarked by Mr. Cayley, we cannot express this fact by less than three equations in integral terms of the coefficients. Thus if the cubic be written

$$a\lambda^3 + 3b\lambda^2 + 3c\lambda + d = 0,$$

we have as one of such ternary systems,

$$U = ac - b^2 = 0, \quad V = bd - c^2 = 0, \quad W = bc - ad = 0.$$

The significant parts of these equations are of course, however capable of being connected by integral multipliers U', V', W', such that

$$U'U + V'V + W'W = 0.$$

Any number of functions U, V, W so related, I call *syzygetic* functions, and U', V', V'' I term the *syzygetic multipliers*.* These in the case supposed are c, a, b, respectively.

In like manner it is evident that the members of any group of functions, more than two in number, whose nullity is implied in the relation of double contact, whether such group form a complete system or not, must be in syzygy.

Thus PQ, PQR, QR, must form a syzygy; nor is there any difficulty in assigning a system of multipliers to exhibit such syzygy. Calling $P = \phi(\lambda)$, $R = \psi(\lambda)$, a and b the two roots of $Q = 0$, I have found that

$$\{(\psi a)^2 + (\psi b)^2\} PQ - (\phi a.\psi a + \phi b.\psi b) PQR + \{(\phi a)^2 + (\psi b)^2\} QR = 0.$$

Again, if we take the *incomplete* system

$$(PQ), \quad (QR), \quad (RP),$$

it will be found that

$$L(QR) + M(RP) + N(PQ) = 0,$$

* There will be in general various such systems of multipliers.

provided that, calling a, b; c, d; e, f, the roots of $P=0$, $Q=0$, $R=0$, respectively, we make

$$L=(k_0+k_1a+k_2a^2+k_3a^3+k_4a^4)\frac{(a-c)(a-d)(a-e)(a-f)}{a-b},$$

$$+(k_0+k_1b+k_2b^2+k_3b^3+k_4b^4)\frac{(b-c)(b-d)(b-e)(b-f)}{b-a};$$

$$M=(k_0+k_1c+k_2c^2+k_3c^3+k_4c^4)\frac{(c-a)(c-b)(c-d)(c-e)(c-f)}{c-d},$$

$$+(k_0+k_1d+k_2d^2+k_3d^3+k_4d^4)\frac{(d-a)(d-b)(d-c)(d-e)(d-f)}{d-c};$$

$$N=(k_0+k_1e+k_2e^2+k_3e^3+k_4e^4)\frac{(e-a)(e-b)(e-c)(e-d)}{e-f},$$

$$+(k_0+k_1f+k_2f^2+k_3f^3+k_4f^4)\frac{(f-a)(f-b)(f-c)(f-d)}{f-e}:$$

k_0, k_1, k_2, k_3, k_4 being quite arbitrary, and L, M, N, although presented in a fractional form, being essentially integral.

This fact of L, M, N constituting a system of multipliers to the syzygy QR, RP, PQ, is easily demonstrated; for

$$QR=(c-e)(c-f)(d-e)(d-f),$$
$$RP=(e-a)(e-b)(f-a)(f-b),$$
$$PQ=(a-c)(a-d)(b-c)(b-d).$$

Hence $\qquad L(QR)+M(RP)+N(PQ)$

$$=(a-c)(a-d)(a-e)(a-f)(b-c)(b-d)(b-e)(b-f)(c-e)(c-f)(d-e)(d-f),$$

$$\times\Sigma\frac{k_0+k_1a+k_2a^2+k_3a^3+k_4a^4}{(a-b)(a-c)(a-b)(a-e)a-f)}=0.$$

My theory of elimination enables me to explain exactly the nature of L, M, N, and the *reason* of their appearance as syzygetic factors.

Let L_r, M_r, N_r signify what L, M, N become, when all the k's except k_r are taken zero. Then the theory given by me in the *Phil. Mag.* for the year 1838, or thereabouts, shews that $L_0\lambda+L_1$ is the *prime derivee* of the first degree between the two equations P and $Q\times R$, or, in other words, will be the remainder integralized of $\frac{Q.R}{P}$.

In like manner $M_0\lambda + M_1$, $N_0\lambda + N_1$ are the integralized remainders of $\frac{R.P}{Q}$ and of $\frac{P.Q}{R}$ respectively.

If now the resultant of P, Q and of Q, R are each zero, but the resultant of P and R is not zero, it will be evident that P, Q, R must be of the form

$$f(\lambda + a)(\lambda + c), \quad g(\lambda + c)(\lambda + d), \quad h(\lambda + d)(\lambda + b);$$

and therefore $P \times R$ will contain Q, and consequently we must have

$$M_0 = 0, \quad M_1 = 0.$$

More generally, if we write

$$\begin{aligned} Q &= 0, \\ \lambda Q &= 0, \\ \lambda^2 Q &= 0, \\ P \times R &= 0, \end{aligned}$$

and eliminate dialytically, *i.e.* treating λ^4, λ^3, λ^2, λ as distinct quantities, we shall obtain*

$$\lambda^4 : \lambda^3 : \lambda^2 : \lambda : 1 :: M_4 : M_3 : M_2 : M_1 : M_0;$$

and therefore when $P \times R$ contains Q,

$$M_0 = 0, \quad M_1 = 0, \quad M_2 = 0, \quad M_3 = 0, \quad M_4 = 0.$$

In like manner, when $Q.P$ contains R,

$$N_0 = 0, \quad N_1 = 0, \quad N_2 = 0, \quad N_3 = 0, \quad N_4 = 0;$$

and when $R.Q$ contains P,

$$L_0 = 0, \quad L_1 = 0, \quad L_2 = 0, \quad L_3 = 0, \quad L_4 = 0.$$

Accordingly, we see from the equation

$$L(QR) + M(RP) + N(PQ) = 0,$$

that if $QR = 0$, $RP = 0$; but PQ *not* $= 0$, then $N = 0$;

---

* This cannot be obtained directly from what is stated in the paper referred to, although contained in the general theory of derivation there given. The arbitrary functions which enter into the expression for the general derivees have been in that paper evaluated only for the prime derivees, which however are only particular phenomena, with reference to the general results of Dialytic Elimination. Hereafter I may give a more general exposition of this remarkable, although ignored or neglected theory. The prime derivees of fx and $f'x$ are Sturm's Functions, cleared of quadratic factors, and are expressed by virtue of the general theorems there laid down as functions of x and of symmetrical functions of the roots of fx.

and therefore $N_0 = 0$, $N_1 = 0$, $N_2 = 0$, $N_3 = 0$, $N_4 = 0$,

and so in like manner for the remaining corresponding two suppositions.*

Before proceeding to consider the remaining case of the highest species of contact, I must observe that besides the equations involved in the condition that $A, B, C; F, G, H$, or, which is the same thing, that any three of them shall all have a factor in common, we must have $\square(U + \lambda V)$ containing the square of such common factor. In the Memoir before adverted to a general theorem will be given and proved, which shews that this latter condition is involved in the former one; in fact, more generally (but still only as a particular case) that when U and V are quadratic functions of n letters, but $U + \epsilon V$ admits of being represented as a complete function of $(n-2)$ quantities only, which are themselves linear functions of the n letters; then $\square(U+\lambda V)$, which is of course a function of λ of the n^{th} degree, will contain the factor $(\lambda - \epsilon)^2$.

When the two conics have four consecutive points in common, the characters of double-point contact and of contact in three consecutive points must exist simultaneously; and consequently the factor common to $A, B, C; F, G, H$, will enter not as a binary but as a ternary factor into $\square(U+\lambda V)$. This gives the extra condition required. As an example take the two conics,

$$U = \frac{y^2}{1-k} + x^2 - z^2 = 0,$$

$$V = y^2 + x^2 - 2kxz + (2k-1)z^2 = 0,$$

$$U + \lambda V = \left(\frac{1}{1-k} + \lambda\right) y^2 + (1+\lambda)x^2 - \{1 + \lambda(1-2k)\} z^2, \\ - 2k\lambda xz.$$

---

* Since we are able to assign the values of the syzygetic multipliers in the equations

$$L(PQ) + M(QR) + N(RP) = 0,$$
$$L'(PQ) + M'(PQR) + N'(QR) = 0,$$
$$L''(QR) + M''(QRP) + N''(RP) = 0,$$
$$L'''(RP) + M'''(RPQ) + N'''(PQ) = 0,$$

it follows that we may eliminate between these four equations any three of the six quantities (PQ), (PRQ), &c., and thus express any one of them in terms of any two others: this method, however, is not practically convenient. I may probably hereafter return to this subject.

The complete determinant of $U + \lambda V$ is therefore

$$\frac{-1}{1-k}\{1+(1-k)\lambda\}\{(1+\lambda)^2 - 2k\lambda(1+\lambda) + k^2\lambda^2\}$$
$$= -\frac{1}{1-k}\{1+(1-k)\lambda\}^3.$$

A, B, C are the determinants of $U + \lambda V$, when $x = 0$, $y = 0$, $z = 0$, respectively. Thus

$$A = \left(\frac{1}{1-k} + \lambda\right)(1+\lambda),$$

$$B = \left(\frac{1}{1-k} + \lambda\right)\{1 + \lambda(1-2k)\},$$

$$C = k^2\lambda^2 - (1+\lambda)\{1+\lambda(1-2k)\}$$
$$= \lambda^2(1-k)^2 - 2\lambda(1-k) - 1;$$

$\lambda = -\dfrac{1}{1-k}$ makes $A = 0$, $B = 0$, $C = 0$, and the factor $\lambda + \dfrac{1}{1-k}$ enters cubed into $\square(U + \lambda V)$.

Hence the two conics have a contact of the third order.

This is easily verified; for if we pass from general to Cartesian and rectangular coordinates, and make z unity; $U=0$ will represent an ellipse with centre at the origin, eccentricity $\sqrt{k}$, and mean focal distance 1, and $V = 0$ the circle of curvature at the extremity of the axis major.*

I had intended to have added some other remarks connected with the present discussion, and also to have appended an *à posteriori* proof of the propositions relative to the reality and otherwise of the vertices and chordal pairs of intersection which I have, at the commencement of this paper, deduced quite legitimately, but in a manner not at first sight perhaps easily intelligible from the general principles of conjugate forms; but this discussion has run on already to a length so much greater than I had anticipated and than the importance of the inquiry may seem to justify, that I must reserve for a future number of the *Journal* what further matter I may have to communicate concerning it.

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* We have thus discussed all the four cases of biconical contact: for an exactly parallel discussion of the theory of contact of a plane with the curve of double curvature in which two surfaces of the second order intersect, see the paper in the *Philosophical Magazine* for this month, before referred to.

POSTSCRIPT.—As I have alluded to Professor Boole's theorem relative to Linear Transformations, it may be proper to mention my theorem on the subject, which is of a much more general character, and includes Mr. Boole's (so far as it refers to *Quadratic* Functions) as a corollary to a particular case. The demonstration will be given in the forthcoming Memoir above alluded to.

Let U be a quadratic function of any number of letters $x_1, x_2 \dots x_n$, and let any number r of linear equations of the general form

$$ {}_1a_r . x_1 + {}_2a_r . x_2 + \dots\dots + {}_na_r . x_n = 0, $$

be instituted between them: and by means of these equations let U be expressed as a function of any $(n-r)$ of the given letters, say of $x_{r+1}, x_{r+2} \dots x_n$, and let U, so expressed, be called M. Let

$$ {}_1a_r . x_1 + {}_2a_r . x_2 + \dots\dots + {}_na_r . x_n $$

be called L_r. Then the determinant of M in respect to the $(n-r)$ letters above given is equal to the determinant of

$$ U + L_1 . x_{n+1} + L_2 . x_{n+2} + \dots\dots + L_r . x_{n+r}, $$

considered as a function of the $(n+r)$ letters

$$ x_1 x_2 \dots\dots x_{n+r}, $$

divided by the square of the determinant

$$ \left.\begin{array}{l} {}_1a_1,\ {}_2a_1 \dots\dots {}_ra_1 \\ {}_1a_2,\ {}_2a_2 \dots\dots {}_ra_2 \\ \dots\dots\dots\dots\dots\dots \\ {}_1a_r,\ {}_2a_r \dots\dots {}_ra_r \end{array}\right\} $$

This I call the theorem of Diminished Determinants.

If now we have U a function of r letters, and V of r other letters, and V is derived from U by linear transformations, *i.e.* by r equations connecting the $2r$ letters; then, since U may be considered as a function of *all the* $2r$ letters with abortive coefficients for all the terms where any of the second set of r letters enter, we may apply our theorem of diminished determinants to the question so considered, and the result will be found to represent Mr. Boole's theorem in a form rather more general and symmetrical, but substantially identical with that given by Mr. Boole.

Thus suppose $\frac{1}{2}ax^2 + bxy + \frac{1}{2}cy^2$ say P, and $\frac{1}{2}au^2 + \beta uv + \frac{1}{2}\gamma v^2$ say Q, are mutually transformable by virtue of the linear equations

$$ lx + my = \lambda u + \mu v, $$
$$ l'x + m'y = \lambda' u + \mu' v, $$

P may be considered as a function of x, y, u, v, and Q as the value of P, when we eliminate x and y by virtue of the two linear equations

$$ L_1 = lx + my - \lambda u - \mu v = 0, $$
$$ L_2 = l'x + m'y - \lambda' u - \mu' v = 0; $$

we have therefore by our our theorem the determinant of Q equal to the squared reciprocal of the determinant $\left.\begin{array}{l} l,\ m \\ l',\ m' \end{array}\right\}$ multiplied by the determinant

$$ \begin{array}{cccccc} a, & b, & o, & o, & l, & l', \\ b, & c, & o, & o, & m, & m', \\ o, & o, & o, & o, & -\lambda, & -\lambda', \\ o, & o, & o, & o, & -\mu, & -\mu', \\ l, & m, & -\lambda, & -\mu, & o, & o, \\ l', & m', & -\lambda', & -\mu', & o, & o, \end{array} $$

which last determinant is evidently equal to the determinant of P multiplied by the square of the determinant $\begin{smallmatrix}\lambda, & \mu \\ \lambda', & \mu'\end{smallmatrix}$. Whence we see that the determinant of Q divided by the square of $\begin{smallmatrix}\lambda, & \mu \\ \lambda', & \mu'\end{smallmatrix}$, is equal to the determinant of P divided by the square of $\begin{smallmatrix}l, & m \\ l', & m'\end{smallmatrix}$. There is also another way more simple, but less direct, by means of which, the theorem of diminished determinants may be made to yield Mr. Boole's theorem of transformation.* Some unavowed use has been made in the foregoing pages of this former theorem, one of the highest importance in the analytical and geometrical theory of quadratic functions. It has been nearly a year in my possession, and I trust and believe that I am committing no act of involuntary misappropriation in announcing it as a result of my own researches.

26, *Lincoln's Inn Fields, August* 12, 1850.

MATHEMATICAL NOTES.

I.—Zur Theorie der Convergens der Kettenbrüche.

Von C. J. Malmstén.

Die Lehre von der Convergens der Kettenbrüche ist viel weniger bearbeitet als die der unendlichen Reihen. Für diese hat man sehr vollständige Regeln, womit man in den meisten Fällen ganz gut auskömmt, so dass der Zwischenraum sehr eng ist, wo man die Convergens oder Divergens nicht beurtheilen kann.

Was dagegen die Kettenbrüche betrifft, so ist das Verhältniss ein ganz anderes. Die sehr wenigen Regeln, die da sind, berühren hauptsächlich nur solche Kettenbrüche, deren alle Glieder positiv sind; sie sind übrigens auch für diese sehr unvollständig. Von einer Bestimmung der *Divergens* ist kaum je Rede gewesen.

Das erste von den folgenden zwey Theoremen, die diese vorhandenseyenden Lücken wenigstens einigentheils auszufüllen beabsichtigen, ist deswegen bemerkenswerth, dass dieselbe Function

$$n.l(n).l^2(n).l^3(n)....l^p(n),\dagger$$

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* Namely, by considering P and Q as *each* derived from some common function of x, y, u, v, w, means of the equations $L_1=0$, $L_2=0$; the law of Diminished Determinants will then indicate the determinants of P and Q, each under the form of fractions having the *same numerator*, but whose denominators will be $\begin{Bmatrix}\lambda, & \mu \\ \lambda', & \mu'\end{Bmatrix}^2$ and $\begin{Bmatrix}l, & m \\ l', & m'\end{Bmatrix}$ respectively.

† In der Folge bezeichnen wir der Kürze wegen:

$l(n) = \log \text{hyp}.\, n, \quad l^2(n) = l\{l(n)\}, \quad l^3(n) = l\{l^2(n)\}$ *u.s:w.*

die (in den Bertrandschen Regeln,—s. *Liouville's Journal*, tom. VII. p. 35) für die unendlichen Reihen den—so zu sagen—Masstab der Convergens oder Divergens abgibt, auch hier für unendliche Kettenbrüche dieselbe Rolle spielte.

Théorème 1. *Si l'on pose*

$$\phi_p(n) = n.l(n) . l^2(n) . l^3(n) . \dots . l^p(n),$$

et qu' on désigne par δ *une quantité positive (quelque petite qu' elle soit, mais pas infiniment petite), la fraction continue*

$$\cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cfrac{b_4}{a_4 + \&c.}}}};$$

où depuis un certain point tous les a *et* b *sont positifs, sera* convergente, *si*

$$\lim \frac{a_n . a_{n-1}}{b_n} . \{\phi_p(n)\}^2 = \infty,$$

et divergente, *si*

$$\lim \frac{a_n . a_{n-1}}{b_n} . \{\phi_p(n)\}^2 \{l^p(n)\}^\delta = 0.$$

Ex. A l' aide de ce théorème on trouvera facilement, que les fractions

$$\cfrac{1}{2 + \cfrac{2^r}{2 + \cfrac{3^r}{2 + \cfrac{4^r}{2 + \cfrac{5^r}{2 + \&c.}}}}} \quad \text{et} \quad \cfrac{1}{2 + \cfrac{4 \log^r 2}{2 + \cfrac{9 \log^r 3}{2 + \cfrac{16 \log^r 4}{2 + \cfrac{25 \log^r 5}{2 + \&c.}}}}}$$

sont toutes les deux *convergentes* pour $r \leqq 2$, et *divergentes* pour $r > 2$.

Théorème 2. *La fraction continue*

$$\left.\cfrac{b_1}{a_1 - \cfrac{b_2}{a_2 - \cfrac{b_3}{a_3 - \cfrac{b_4}{a_4 - \&c.}}}}\right\} \dots\dots\dots (A),$$

où depuis un certain point tous les a *et* b *sont positifs, sera convergente, s'il y a un tel* ψ_n, *dont la limite pour* $n = \infty$ *est positive et déterminée, que pour* $n = \infty$,

$$\frac{a_n \cdot a_{n-1}}{b_n} - \frac{(1 + \psi_n)(1 + \psi_{n-1})}{\psi_n} \geqq 0.$$

Cor. 2. Supposons $\psi_n = b_n$; il s'ensuit que la fraction (A) est convergente toutes les fois que pour les valeurs très grandes de n

$$a_n \overset{=}{>} 1 + b_n.$$

Cor. 2. Supposons $\psi_n = 1$; la fraction continue (A) est convergente, si pour $n = \infty$

$$\frac{a_n \cdot a_{n-1}}{b_n} \geqq 4.$$

II.—Demonstration of certain Propositions in the Eleventh Section of Newton's Principia.

By W. Walton.

Si corpora duo viribus quibusvis se mutuo trahunt, et interea revolvuntur circa gravitatis centrum commune; dico quod figuris, quas corpora sic mota describunt circum se mutuò, potest figura similis et æqualis, circum corpus alterutrum immotum, viribus iisdem describi.—Newton, *Principia*, Prop. LVIII. Theor. XXI. Lib. I.

The demonstration of this proposition, which has been given by Newton, and which has been adopted in modern editions of selections from his *Principia*, may I think be advantageously replaced by the slightly different one here proposed, which is based upon a proposition of the second section.

Upon each of the two bodies P and S impress a force, in the direction PS, equal to the attraction of P upon S, and also impress upon these two bodies two equal and parallel motions of such magnitude and direction as, in conjunction with the force just mentioned, to destroy all motion in S. These operations will not affect the relative motion of P and S.

Thus P may be regarded as moving round S as a fixed centre of attraction, of which the absolute force is equal to $P + S$. Hence, PV denoting the chord of curvature at P of this relative orbit, through S, and SY the perpendicular

from S upon its tangent at P, the orbit may be any whatever which is consistent with the equation

$$\frac{2h^2}{SY^2.PV} = (P+S).f(PS) \dots\dots\dots\dots (1),$$

as we know by the second Section of the *Principia*, h denoting twice the radial area swept out in a unit of time in the relative orbit, and $f(PS)$ a function of PS dependent upon the law of the force.

If the body S were absolutely fixed, P might describe any orbit whatever which is consistent with the equation

$$\frac{2h'^2}{SY^2.PV} = S.f(PS) \dots\dots\dots\dots (2).$$

Now, if
$$\frac{h^2}{P+S} = \frac{h'^2}{S} \dots\dots\dots\dots (3),$$

the equations (1) and (2) become identical. The equation (3) therefore is a sufficient and necessary condition for the *possibility* of the relative orbit of P round S in motion being made to coincide in every respect with that of P round S fixed.

Since $\quad h = V.SY, \;$ and $\; h' = V'.SY,$

supposing the orbits similar and equal, the equation (3) becomes

$$\frac{V^2}{P+S} = \frac{V'^2}{S},$$

or
$$V : V' :: (P+S)^{\frac{1}{2}} : S^{\frac{1}{2}}.$$

Cor. 1. Since the curves are equal and similar in the two cases, it is plain that T, T', denoting the periodic times in the former and latter curves respectively,

$$T : T' :: h' : h;$$

hence, by (1), $\quad T : T' :: S^{\frac{1}{2}} : (P+S)^{\frac{1}{2}}.$

This proportion is asserted in Prop. LIX. Theor. XXII.

Cor. 2. S being reduced to rest in the manner described above, P may be regarded as acted on by a force

$$\frac{P+S}{PS^2},$$

tending towards S. The periodic time of P round S will therefore be

$$\frac{2\pi a^{\frac{3}{2}}}{(P+S)^{\frac{1}{2}}},$$

a being the semi-axis major of P's orbit round S.

If S were absolutely fixed, then, a' denoting the semi-axis major of P's orbit, the periodic time of P round S would be equal to

$$\frac{2\pi a'^{\frac{3}{2}}}{S^{\frac{1}{2}}}.$$

If, instead of supposing the orbits equal and similar, we assume the periodic times to be equal,

$$\frac{2\pi a^{\frac{3}{2}}}{(P+S)^{\frac{1}{2}}} = \frac{2\pi a'^{\frac{3}{2}}}{S^{\frac{1}{2}}},$$

or $$a : a' :: (P+S)^{\frac{1}{3}} : S^{\frac{1}{3}},$$

or, if K denote the first of the two mean proportionals between $P+S$ and S,

$$a : a' :: P+S : K;$$

which is the proportion asserted in Prop. LX. Theor. XXIII.

Cambridge, Oct. 2, 1849.

III.—Reduction of the General Equation of the Second Degree.

The form of this equation, when complete in all its terms, is

$$Ax^2+A'y^2+A''z^2+2Byz+2B'xz+2B''xy+2Cx+2C'y+2C''z+E=0;$$

but as we wish to remove the part containing the products of the variables, we may limit our attention to that portion of the left-hand member which is of the second degree, viz.

$$Ax^2 + A'y^2 + A''z^2 + 2Byz + 2B'xz + 2B''xy \ldots.(1).$$

This may always be put under the form

$$(ax+by+cz)^2 + (a'x+b'y+c'z)^2 + (a''x+b''y+c''z)^2 \ldots(2);$$

for if we seek the conditions that (1) and (2) may be identical, we get

$$\left.\begin{array}{ll} a^2+a'^2+a''^2 = A, & bc+b'c'+b''c'' = B \\ b^2+b'^2+b''^2 = A', & ac+a'c'+a''c'' = B' \\ c^2+c'^2+c''^2 = A'', & ab+a'b'+a''b'' = B'' \end{array}\right\} \ldots(3),$$

which are six equations connecting nine indeterminate quantities. Hence we may put (1) under the form (2) in an

infinite number of ways. Now, through the origin draw three planes, whose direction-cosines are proportional respectively to (a, b, c), (a', b', c'), (a'', b'', c''); then the expressions

$$ax + by + cz, \quad a'x + b'y + c'z, \quad a''x + b''y + c''z,$$

are proportional to the perpendiculars from the point (x, y, z) on these planes. If we denote these perpendiculars by p_1, p_2, p_3; (2) will assume the form

$$Dp_1^2 + D'p_2^2 + D''p_3^2 \;\ldots\ldots\ldots\ldots\; (4).$$

We have thus got rid of x, y, z, and in their place introduced perpendiculars on three fixed planes. If we choose these planes as new coordinate planes, and suppose x', y', z' to be the coordinates of a point referred to their lines of intersection as axes, x', y', z' will evidently be proportional to p_1, p_2, p_3, and (4) may be written

$$Kx'^2 + K'y'^2 + K''z'^2 \;\ldots\ldots\ldots\ldots (5).$$

We have thus shewn that by simple transformation of coordinates, we can always reduce (2) to the form (5), and also that there are an infinite number of sets of planes, which when chosen as coordinate planes will effect this change.

We will now proceed to find, if among these sets of coordinate planes there is one which is rectangular. Let (l, m, n), (l', m', n'), (l'', m'', n'') be the direction-cosines of the three planes of one set. Then we shall have

$$\begin{array}{lll} a = kl, & b = km, & c = kn, \\ a' = k'l', & b' = k'm', & c' = k'n', \\ a'' = k''l'', & b'' = k''m'', & c'' = k''n'', \end{array}$$

k, k', k'' being constants to be determined. The conditions (3), together with those expressing the perpendicularity of the planes, and those connecting the direction-cosines, will now become

$$\begin{array}{l} k^2l^2 + k'^2l'^2 + k''^2l''^2 = A \ldots\ldots (\alpha), \\ k^2m^2 + k'^2m'^2 + k''^2m''^2 = A' \ldots\ldots (\beta), \\ k^2n^2 + k'^2n'^2 + k''^2n''^2 = A'' \ldots\ldots (\gamma), \\ k^2mn + k'^2m'n' + k''^2m''n'' = B \ldots\ldots (\delta), \\ k^2ln + k'^2l'n' + k''^2l''n'' = B' \ldots\ldots (\epsilon), \\ k^2lm + k'^2l'm' + k''^2l''m'' = B'' \ldots\ldots (\zeta), \end{array}$$

$$ll' + mm' + nn' = 0, \quad l^2 + m^2 + n^2 = 1,$$
$$ll'' + mm'' + nn'' = 0, \quad l'^2 + m'^2 + n'^2 = 1,$$
$$l'l'' + m'm'' + n'n'' = 0, \quad l''^2 + m''^2 + n''^2 = 1,$$

which are twelve equations between twelve quantities; so that unless some of these equations are derivable one from the other, the quantities can be all determined. It remains to prove that their values are possible.

(1) Multiply (α) by l, (ϵ) by n, (ζ) by m, and add. Therefore
$$(k^2 - A)\,l - B''m - B'n = 0 \,\ldots\ldots\ldots (6).$$

(2) Multiply (β) by m, (δ) by n, (ζ) by l, and add. Therefore
$$B''l - (k^2 - A')\,m + Bn = 0 \ldots\ldots\ldots\ldots (7).$$

(3) Multiply (γ) by n, (δ) by m, (ϵ) by l, and add. Therefore
$$B'l + Bm - (k^2 - A'')\,n = 0 \,\ldots\ldots\ldots (8).$$

Eliminating l, m, n between (6), (7), (8), by cross multiplication, we get

$$(k^2 - A)(k^2 - A')(k^2 - A'') \\ - B^2(k^2 - A) - B'^2(k^2 - A') - B''^2(k^2 - A'') - 2BB'B'' = 0,$$

the well-known discriminating cubic, the roots of which are always possible; and we infer from the symmetry of the equation, that the three roots are the three unknown quantities k^2, k'^2, k''^2.

Expression (4) will now become

$$k^2p_1^2 + k'^2p_2^2 + k''^2p_3^2;$$

and when the new planes are taken as coordinate planes, p_1, p_2, p_3 will be equal to x, y, z, respectively. Therefore (5) becomes
$$k^2x'^2 + k'^2y'^2 + k''^2z'^2.$$

We thus see that we can always find one set of rectangular, or an infinite number of oblique axes, when referred to which the equation will assume the form

$$k^2x^2 + k'^2y^2 + k''z^2 + 2Cx + 2C'y + 2C''z + E = 0.$$

W. J. S.

END OF VOL. V.

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